SOME LIKELIHOOD RATIO TYPE TESTS FOR EXPONENTIAL DISTRIBUTION WITH RESTRICTED ALTERNATIVES

By

Bassam Suleiman Abu-Amra

Yarmouk University

Department of Statistics

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Thesis Defence Committee

Dr. Hussein M. Al-Rawwash	Chairman	
Dr. Mohammed S. Ahmed	Member	
Dr. Walid A. Abu-Dayyah	Member	

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TO MY PARENTS AND MY FAMILY

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Bassam S. Abu-Amra

اختبار بعض النسب الاحتمالية لتوزيع دالة القوى في حالة البدائل المقيدة

الملخص

هذه الرسالة تتناول ثلاثة اختبارات تعتمد على دالة النسبة الاحتمالية وهي اختبار النسبة الاحتمالية المعروفة (L.R.T.)، اختبار النسبة الاحتمالية المعدلة (Modified L.R.T.) واختبار النسبة الاحتمالية المعتمدة على تقلص المسافة (M.D.L.R.T.). وقد تم ايجاد فعالية هذه الاختبارات لفرضيات بديلة مقيدة في حالة توزيع دالة القوى الثنائية. وقد استخدمت الفرضية الاساسية الممثلة في معالم هذا التوزيع وتساوي النقطة (1,1) في حين ان الفرضية البديلة هي عبارة عن معالم هذا التوزيع موجودة في مخروط منزوع منه النقطة (1,1) وله زاوية مقدارها β*، وهذا المخروط راسه في النقطة (1,1) في المستوى الديكارتي.

ففي الجزء الاول، هناك مقدمة لمجموعة اعمال سابقة في نفس موضوع الرسالة، واهمها الاعمال المتعلقة بأيتان Eaton (1970) وبيرنباوم Birnbaum (1955) التي لها ارتباط بالاختبارات السابقة حيث انها الافضل (Admissable Tests) اذا تحققت شروط معينة.

في الجزء الثاني ، يوجد اشتقاق كامل لإقتران الفعالية (Power function) للإختبارات السابقة .

في الجزء الثالث ، تم اثبات ان منطقة القبول لبعض الإختبارات السابقة هي مجموعات مقعرة (convex sets) ومتناقصة (V-decreasing) ولهذا فهي اختبارات افضل . وهناك خواص متعلقة باقتران الفعالية حيث انه متزايد على الفترة $[0,\bar{\beta}]$ ومتناقص على فترة اخرى $[\bar{\beta},\beta^*]$ ، وله تماثل حول $\bar{\beta}$ ايضا .

في الجزء الأخير من هذه الرسالة ، توجد مقارنات حسابية بين اقترانات الفعالية المختلفة المعتمدة على فرضيات بديلة ، مختلفة حسب اختلاف زاوية المخروط والزوايا هي $2\pi, \pi/2, \pi/6, \pi/3, \pi/4$ وهذه المقارنات تمت على مستوى دلالة 0.05 ،ومن خلال المقارنات ، لوحظ انه كلما كانت الفرضية البديلة مقيدة اكثر ، كلما ادى الى ازدياد اقتران الفعالية ، اي كلما قلت زاوية المخروط سنحصل على اقتران فعالية افضل .

ABSTRACT

The purpose of this thesis is to study the tests of statistical hypotheses with restricted alternatives in a bivariate exponential distribution. It considers, in particular, the Likelihood Ratio Test (L.R.T.) and two other tests called the Minimum Distance Likelihood Ratio Test (M.D.L.R.T.) and the Modified L.R.T. The restricted alternatives considered here, are some closed convex cones with vertex at (1,1) and angle β^*.

This thesis consists of two parts. The first part contains derivation of the three tests, and their power functions for different types of restrictions, Furthermore, some monotonicity and symmetry properties of the power functions are partially proved. Also, using the results of Birnbaum (1955) and under (1970), and for certain restrictions, it is proved that the L.R.T., M.D.L.R.T. and Modified L.R.T are admissible tests.

The second part contains some numerical computations of the power function for the different tests and for cone angles $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$ and 2π. The tables of the power function suggest some remarks on the monotonicty and dominations of these tests.

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CHAPTER ONE

TESTING HYPOTHESES UNDER RESTRICTED ALTERNATIVES

1.1 Introduction.

Consider the bivariate independent exponential random vector (X,Y) with parameters (θ_1, θ_2) having probability density function (pdf) of the form $f(x,y;\theta_1,\theta_2)$ given by

$$f(x,y;\theta_1,\theta_2) = \frac{1}{\theta_1\theta_2} e^{-x/\theta_1 - y/\theta_2} \quad , \; x,y > o \qquad (1.1.1)$$

$$= 0 \qquad\qquad , \text{ otherwise },$$

where $(\theta_1,\theta_2) \in \Omega$, the parameter space, which is the non-negative quadrant.

Consider the testing problem

$$H_0 : (\theta_1,\theta_2) = (\theta_{10}, \theta_{20})$$

against (1.1.2)

$$H_1 : (\theta_1,\theta_2) \in V \setminus \{(\theta_{10}, \theta_{20})\}$$

where $(\theta_{10}, \theta_{20})$ are given and

$$V = \left\{(x,y) : x \geq \theta_{10}, \; \theta_{20} \leq y \leq (x-\theta_{10}) \tan \beta^* + \theta_{20}\right\}$$

is a closed convex cone in $\mathbb{R}^2$ with vertex at $(\theta_{10}, \theta_{20})$ and angle β^* satisfying $0 \leq \beta^* \leq \pi/2$. Note that for $\beta^* = 2\pi$, the cone V will reduce to the whole parameter space.

Without loss of generality, assume that $(\theta_{10},\theta_{20})=(1,1)$. Therefore the testing problem (1.1.2) will become

$$H_0 : (\theta_1, \theta_2) = (1,1)$$

against (1.1.3)

$$H_1 : (\theta_1, \theta_2) \in V \setminus \{(1,1)\}$$

Also, the cone V become

$$V = \left\{(x,y) : x \geq 1, 1 \leq y \leq (x-1) \tan \beta^* + 1\right\}.$$

Exponential distribution plays a vital role and is often proposed for modeling the lifetime distribution of items of complex equipments such as electronic components, light bulbs and many others. An example of restricted alternative is the ordered altrnatives for which the rank order of the means is given. In some cases the qualitative characteristics can be ranked but not easily measured, therefore, we must deal with restricted alternatives. A real life example of restricted alternatives can be formulated as follows. Suppose we have two factories that produce light bulbs ranked in terms of average life for their products. The null hypothesis is that the bulbs produced from the two factories have the same average life, and the alternative may be that the bulbs produced by the first factory have live twice the life of the bulbs produced by the second factory. This testing problem is of the form (1.1.2) with certain cone.

In this thesis, the Likelihood Ratio Test (L.R.T.) for the testing problem given by (1.1.2) and its power function are obtained. In addition, we have two other tests called

Minimum Distance Likelihood Ratio Test (M.D.L.R.T.) and Modified L.R.T. Also, the power function of these tests are obtained. Furthermore, we give some numerical comparison between the power function of the tests.

In the following sections of this chapter, some mathematical preliminaries are given in section 1.2. Section 1.3 contains a review of the literature related to the subject of this thesis. Section 1.4 summarizes the contents of the thesis.

1.2 Mathematical Preliminaries

Some definitions and theorems that are needed in the thesis are listed below.

Definition (1.2.1)

A set $G \subset \mathbb{R}^2$ is said to be convex if for any two points x,y in G, the point $\alpha x + (1-\alpha)y$ is also in G, for all α such that $0 \le \alpha \le 1$.

Notice that $\{\alpha x + (1-\alpha)y : 0 \le \alpha \le 1\}$ is the line segment joining x and y. Therefore, G is convex if the line segment between any two points in G is a subset of G.

Definition (1.2.2)

Consider the problem of testing the hypothesis

$$H_0 : \theta \in \Theta_0 \quad \text{against} \quad H_1 \; \theta \in \Theta_1$$

where Θ_0 and Θ_1 are disjoint subsets of Θ. A test ϕ_1 is said to dominate another test ϕ_0 if the following two conditions hold:

i. $E_\theta \phi_1 \leq E_\theta \phi_0 \quad \forall \theta \in \theta_0$

ii. $E_\theta \phi_1 \geq E_\theta \phi_0 \quad \forall \theta \in \theta_1$

Also, ϕ_1 strictly dominates ϕ_0 if one of the above inequalities is strict for some $\theta' \in \theta$.

Definition (1.2.3)

A class $\mathscr{C}$ of tests, is said to be complete, if, given any test ϕ not in $\mathscr{C}$, there exists a test ϕ_0 in $\mathscr{C}$ which dominates ϕ. If $\mathscr{C}$ contains no subclass which is complete, then $\mathscr{C}$ is said to be minimal complete.

Definition (1.2.4)

A test ϕ_1 is said to be admissible if there exists no test which strictly dominate it, otherwise it is said to be inadmissible.

Definition (1.2.5)

Let V be a cone with vertex at (0,0). The cone V^o is said to be the dual cone of V, if it can be written in the form

$$V^o = \left\{ y \in R^2 : y.v \leq 0, \text{ for all } v \in V \right\}$$

In some situations, V^o is called the polar cone or the negative conjugate cone of V.

Definition (1.2.6)

For a given closed convex cone V, define an order relation between points in $\mathbb{R}^2$ as follows

Let $x, y \in \mathbb{R}^2$ then

$$x \le [V]\ y \quad \text{if } x - y \in V^o - \{0\}$$

where V^o is the dual cone of V.

Definition (1.2.7)

A set $G \subset \mathbb{R}^2$ is said to be a decreasing set with respect to cone V or V-decreasing, if for any $x, y \in \mathbb{R}^2$, such that $y \in G$ and $x < [V]\ y$ then $x \in G$.

Definition (1.2.8)

Let $f(x,\theta)$ be pdf such that $x \in \mathbb{R}^n$, $\theta \in \Omega \in \mathbb{R}^n$ where Ω is an open set. Let Θ_o be an open subset of Ω. So, an estimator $\hat{\hat{\theta}}$ of θ is called a minimum distance estimator (M.D.E.) restricted to Θ_o if it minimizes

$$\| x-\theta \|^2 \quad \text{for} \quad \theta \in \Theta_o.$$

i.e. $\hat{\hat{\theta}}$ is the projection of the vector x onto the region Θ_o.

Definition (1.2.9)

Let $f(x,\theta)$ be pdf, and we are given the testing problem

$$H_o : \theta \in \Theta_o \quad \text{against} \quad H_1 : \theta \in \Theta_1$$

where Θ_o and Θ_1 are disjoint subsets of Θ. Define the Minimum Distance Likelihood Ratio Test (M.D.L.R.T.) as:

$$\phi(x) = \begin{cases} 0 \quad , & \Lambda \geq k_o \\ 1 \quad , & \Lambda < k_o \end{cases}$$

where $$\Lambda = \frac{f(x,\hat{\hat{\theta}}_o)}{f(x,\hat{\hat{\theta}}_1)}$$

and $\hat{\hat{\theta}}_i$ is the M.D.E. restricted to Θ_i; $i = 0, 1$.

<u>Theorem (1.2.1) Birnbaum (1955)</u>

Consider a random variable X which has the pdf of the form

$$f(x,\theta) = A(x)\ B(\theta)\ e^{x.\theta} \quad , \quad x \in \mathbb{R}^n \ , \ \theta \in \Omega \subset \mathbb{R}^n.$$

For the testing problem

$H_o : \theta = \theta_o$ against $H_1 : \theta \neq \theta_o$,

where $\theta_o \in Int(\Omega)$,

the class of all tests which have convex acceptance region form a complete class. For the case that Ω contains spheres of arbitrary large radii, the class is a minimal complete class.

<u>Theorem (1.2.2) Eaton (1970)</u>

Consider the setup of the Birnbaum theorem. Let V be a closed convex cone with vertex (0,0) in $\mathbb{R}^n$. Then for testing

$H_o : \theta = 0$ against $H_1 : \theta \in V \setminus \{0\}$,

the class of all tests which has convex and V-decreasing acceptance region form a minimal complete class.

1.3 Review of Literature

In this section, we present some of the previous work related to the testing problem of the exponential distribution based on restricted or unrestricted alternative space.

Many authors have worked on this type of testing problems. Most of the work involved is about testing location or scale parameters under different considerations.

Consider a random vector $X = (X_1,\ldots,X_k)$, which has a probability density function of the form

$$f(x, \underline{\beta}, \underline{\sigma}) = \prod_{i=1}^{k} \frac{1}{\sigma_i} \exp\left[-\sum_{i=1}^{k} \frac{1}{\sigma_i}(x_i-\beta_i)\right], \quad x_i \geq \beta_i, \sigma_i > 0, i=1,\ldots,k$$

$$= 0 \qquad , \qquad \text{elsewhere} \qquad (1.3.1)$$

where $\underline{\beta} = (\beta_1,\ldots,\beta_k)'$ and $\underline{\sigma} = (\sigma_1,\ldots,\sigma_k)'$ are the vector of location and scale parameters respectively.

N. Singh (1983) considered the case that $\sigma_i = \sigma$ $\forall$ $i=1,\ldots,k$ and the following testing problem:

H_0 : $\beta_1 = \beta_2 = \ldots = \beta_k = \beta$, β is unspecified, σ unknown

against

H_1 : at least two of β's are unequal.

He derived the likelihood ratio test, and showed that the L.R.T. reduces to an equivalent test based on a statistic that has an F-distribution.

P. G. Wong (1978) considered the case k = 1 and the testing problem

$$H_o : \beta = \beta_o \quad \text{vs.} \quad H_1 : \beta > \beta_0$$

or

$$H_o : \beta = \beta_o \quad \text{vs.} \quad H_1 : \beta < \beta_0$$

where β_o is a given parametric value.

Let the statistic $Q_1 = \dfrac{y_n}{y_1}$, where y_1, y_n are the 1-st and the n-th order statistics of the random sample. Q_1 is called the extremal quotient. The distribution of Q_1 is scale free. He developed a procedure based on the extremal quotient for testing the location parameter of the exponential distribution.

Paulson (1941) considered the cases k = 1 or k = 2 for the following testing problems:

H_o : $\beta_1 = 0$ when σ_1 is known, let $\sigma_1 = 1$

H_o : $\beta_1 = 0$ when σ_1 is unknown.

H_o : $\beta_1 = \beta_2$ when $\sigma_1 = \sigma_2$.

He derived the L.R.T. and the power function of the testing problems above. These tests are shown to be completely unbiased.

Epstein and Tsao (1983) considered the case k=2 and the following hypotheses :

H_{01} : $\sigma_1 = \sigma_2$ (Assuming that β_1, β_2 are known)

H_{02} : $\sigma_1 = \sigma_2$ (Assuming that $\beta_1 = \beta_2 = \beta_0$, where β_0 is unknown)

$H_{03} : \sigma_1 = \sigma_2$

$H_{04} : \beta_1 = \beta_2$ (Assuming that σ_1, σ_2 are known)

$H_{05} : \beta_1 = \beta_2$ (Assuming that $\sigma_1 = \sigma_2 = \sigma_0$ where σ_0 is unknown)

$H_{06} : \beta_1 = \beta_2$

$H_{07} : \beta_1 = \beta_2$ and $\sigma_1 = \sigma_2$.

They derived the L.R.T. based on the smallest observations of the random samples from pdf given by (1.3.1) with $k = 2$. These likelihood ratio tests can be reduced to equivalent tests which are distributed as χ^2 or F distribution, though the authors did not succeed in reducing the L.R.T. for the hypothesis H_{07} to some known criteria as F or χ^2 distribution.

Marcus (1976) was interested in testing

$$H_0 : \beta_1 = \beta_2 = \ldots = \beta_k$$

against

$$H_1 : \beta_1 \leq \beta_2 \leq \ldots \leq \beta_k$$ with at least one strict inequality.

where $\sigma_i = \sigma \ \forall\ i=1,2,\ldots,k$.

By using Monte Carlo Sampling, a comparison among the power functions of four test statistic is made. These test statistics are the likelihood ratio statistic (λ), Kolmogorov-Smirnov statistic (D^+), and other two statistics U, Q which are cumulative functions of ordered uniform deviates and their logarithms respectively. He concluded that if the inequalities are among the relatively small means then the λ and Q tests are more powerful than the other two, whereas if the inequalities are among the relatively high means then the reverse is true.

P.B. Nagarsenker and B.N. Nagarsenker (1986) were interested in testing the null hypothesis $H_o : \sigma_1 = \sigma_2 = \ldots = \sigma_k$, the β_i's $(i=1,2,\ldots,k)$ are unspecified against the general alternatives.

In case of two exponential populations, the distribution of the L.R. statistic for testing H_o, has been considered by Paulson (1941), Epstein and Tsao (1953), and others. For more than two exponential populations, no exact distribution of the L.R. statistic is available in a closed form. They obtained the exact distribution of the L.R. statistic for testing H_o in a computational form for the case of equal sample sizes. This distribution was used to compute the exact significance points of the test statistic.

Now, since the normal and exponential populations belong to the exponential family of distributions, we thought that it is interesting to present some of the previous work related to the testing problem in case of multivariate normal distribution.

Consider the following testing problem for multivariate normal distribution with mean vector η; $\eta'=(\eta_1,\eta_2,\ldots,\eta_p)$ and identity covariance matrix.

$$H_o : (\eta_1,\ldots,\eta_p) = (0,\ldots,0)$$

against (1.3.2)

$$H_1 : (\eta_1,\ldots,\eta_p) \in V^* \setminus \{(0,0)\}$$

where $V^* = \left\{(r,\beta)) : r \geq 0 \ , \ 0 \leq \beta \leq \beta^*\right\}$ is a closed convex cone with angle β^* such that β^* satisfies either $\beta^* = 2\pi$ or $0 \leq \beta^* \leq \pi$.

Bartholomew, Kudô, Nüesch, Shirahata, Marden, Al-Rawwash, and others have considered the testing problem given by (1.3.2).

Bartholomew (1959a,b) was interested in testing

$$H_o : \eta_1 = \eta_2 = \ldots = \eta_k$$

against

$$H_1 : \eta_1 \geq \eta_2 \geq \ldots \geq \eta_k$$

He obtained the L.R.T. and its null distribution, also, he showed that the L.R. statistic has distribution which depends on the χ^2-distribution and certain probabilities.

Bartholomew (1961) obtained the power function of the L.R.T. under some special cases. He observed through his computations that the L.R.T. for the restricted alternative dominates the usual χ^2-test in terms of the power function.

Kudô (1963) is concerned with a testing problem of the form

$H_o : \eta_i = 0$ $(i=1,\dots,k)$ against $H_1 : \eta_i \geq 0$ for $i=1,2,\dots,k$ where the inequality is strict for at least one value of i. He derived the likelihood ratio criterion and obtained its null distribution.

Al-Rawwash (1986) partially proved a conjecture about the L.R.T.. The conjecture says that the more restrictions put on the alternative space, the power function of L.R.T. increases. Particularly, in a bivariate normal distribution, he showed that the L.R.T. ϕ_{V_1} dominate the L.R.T. ϕ_{V_2}, $V_1 \subset V_2$ where ϕ_V is the L.R.T. for testing

$$H_o : (\eta_1,\eta_2) = (0,0) \quad \text{vs.} \quad H_1 : (\eta_1,\eta_2) \in V \setminus \{(0,0)\}$$

He proved the above conjecture for the following cases:

1. $V_1 = \mathbb{R}^{+2}$, $V_2 = \mathbb{R}^2$, where $\mathbb{R}^+ = [0,\infty)$
2. $V_1 = \mathbb{R}^+ \times \mathbb{R}$, $V_2 = \mathbb{R}^2$
3. $V_1 = \mathbb{R}^{+2}$, $V_2 = \mathbb{R}^+ \times \mathbb{R}$

In this thesis, we derive the L.R.T., M.D.L.R.T. and Modified L.R.T. of the testing problem (1.1.3). In addition, we obtain their power functions. Also, we observe, from tables of the power functions, some properties of these tests such as admisibility , symmetry, and monotonicity.

1.4 Summary of the thesis

This thesis consists of four chapters. In the first chapter, we give the statistical model and the testing problem under consideration. Furthermore, we present some mathematical preliminaries and review of important literature related to our testing problem.

In the second chapter, we derive the L.R.T., M.D.L.R.T. and Modified L.R.T. and their power functions under two types of alternative spaces. The first is the whole parameter space and the second is a cone with vertex at (1,1) and angle $\beta^* \le \pi / 2$.

In the third chapter, we present some properties of these tests such as admissibility, monotonicity and symmetry of the power functions, for the two types of alternative spaces.

In the last chapter, we give some numerical comparison between the different power functions of the L.R.T., M.D.L.R.T. and Modified L.R.T. for cone angles $\beta^* = \pi/6$, $\pi/4$, $\pi/3$, $\pi/2$ and 2π.

CHAPTER TWO

DERIVATION OF THE L.R.T., M.D.L.R.T. AND MODIFIED L.R.T. AND THEIR POWER FUNCTIONS

In the following two sections, we derive the L.R.T., M.D.L.R.T. and Modified L.R.T. and their power functions for the testing problem given by (1.1.2).

2.1 Derivation of the L.R.T., M.D.L.R.T and Modified L.R.T.

Consider the bivariate independent exponential distribution with parameters (θ_1,θ_2). It is desired to test the following problem:

$$P(V) \qquad H_o : (\theta_1,\theta_2) = (1,1)$$

against (2.1.1)

$$H_1 : (\theta_1,\theta_2) \in V \setminus \{(1,1)\}$$

where V is some closed convex cone, which has the form

$$V_{\beta^*} = \left\{(x,y) : x \geq 1,\ 1 \leq y \leq b(x-1) + 1 \ ,\ b = \tan\beta^* \right\}$$

such that $0 \leq \beta^* \leq \pi/2$ or $\beta^* = 2\pi$. For simplicity we use V_1 for a cone with angle $\beta^* \in (0, \pi/2)$ and V_2 for the cone with angle 2π.

In order to derive the M.D.L.R.T., we obtain first the M.D.E. $(\hat{\hat{\theta}}_1,\hat{\hat{\theta}}_2)$ restricted to the cone V_1 which is the alternative space of the testing problem (2.1.1).

Lemma (2.1.1) The M.D.E. of (θ_1,θ_2) restricted to the cone V_1 is given by

$$(\hat{\hat{\theta}}_1,\hat{\hat{\theta}}_2)=\begin{cases}(x,y) & \text{if } (x,y)\in V_1\\ (1,1) & \text{if } (x,y)\in V_1^{o}\\ \left(\dfrac{x+by-b+b^2}{b^2+1},\ \dfrac{bx+b^2y-b+1}{b^2+1}\right) & \text{if } (x,y)\in V_1^{+}\\ (x,1) & \text{if } (x,y)\in V_1^{-}\end{cases} \qquad (2.1.2)$$

where $b = \tan\beta^*$ and V_1, V_1^o, V_1^+, V_1^- are as illustrated in Figure 2-1.

<u>Proof</u>:

To obtain the M.D.E. $(\hat{\hat{\theta}}_1,\hat{\hat{\theta}}_2)$ restricted to V_1, partition the space R^2 into 4 regions V_1, V_1^+, V_1^o and V_1^- which are illustrated in Figure 2-1:

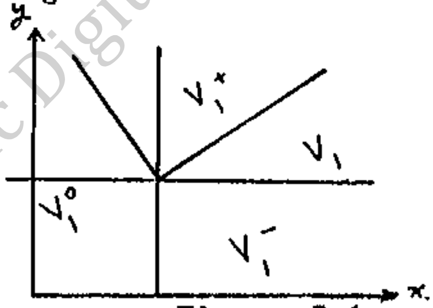

Figure 2-1

Now, if $(x,y)\in V_1$ then $(\hat{\hat{\theta}}_1,\hat{\hat{\theta}}_2) = (x,y)$. But if $(x,y) \notin V_1$, then the M.D.E. of (θ_1,θ_2) will be the projection of (x,y) on the boundary of V_1. We have 3 cases:

1. For $(x,y) \in V_1^+$, the M.D.E. is in the set

$$\{(u,w)\ :\ u = b(w-1)+1\ ,\ u\geq 1,\ w\geq 1\}$$

Thus $\theta_2 = b(\theta_1-1) + 1$

Let D be the distance between (x,y) and the point (θ_1,θ_2).

i.e., $D^2 = (x-\theta_1)^2 + (y-\theta_2)^2$

$$= (x-\theta_1)^2 + (y-b\theta_1-1+b)^2$$

so that, $\frac{\partial D^2}{\partial \theta_1} = -2(x-\theta_1) - 2b(y-b\theta_1-1+b) = 0$

$$\Rightarrow (x-\theta_1) + b(y-b\theta_1-1+b) = 0$$

$$\Rightarrow (x+by) - \theta_1(b^2+1) + b^2+b = 0$$

This gives,

$$\hat{\hat{\theta}}_1 = \frac{x+by-b+b^2}{b^2+1}$$

and $$\hat{\hat{\theta}}_2 = b\,\hat{\theta}_1 - b + 1$$

$$= \frac{b(x+by-b+b^2)}{b^2+1} - b + 1$$

$$= \frac{bx+b^2y-b+1}{b^2+1}$$

2. For $(x,y) \in V_1^o$, then $(\hat{\hat{\theta}}_1,\hat{\hat{\theta}}_2) = (1,1)$

3. For $(x,y) \in V_1^-$, the M.D.E. is in the set $\{(u,1) : u \geq 1\}$. It is a subset of the boundary of the cone V_1. So that $(\hat{\hat{\theta}}_1,\hat{\hat{\theta}}_2) = (x,1)$ is the closest point in this set to (x,y). This completes the proof.

Now, we obtain the M.D.L.R.T.

Theorem (2.1.2)

Consider the bivariate independent exponential distribution with parameters (θ_1, θ_2). The M.D.L.R.T. ϕ_{V_1} is of the form

$$\phi_{V_1} = \begin{cases} 0 & \Lambda \geq k_1 \\ 1 & \Lambda < k_1 \end{cases} \qquad (2.1.3)$$

where k_1 is a constant based on the level of significance, and

$$\Lambda = \begin{cases} x\, y\, e^{-x-y+2} & \text{if } (x,y) \in V_1 \\ 1 & \text{if } (x,y) \in V_1^0 \\ \left[\dfrac{x+by-b+b^2}{b^2+1}\right] e^{-xt} \left[\dfrac{bx+b^2y-b+1}{b^2+1}\right] e^{-ys} & \text{if } (x,y) \in V_1^+ \\ x\, e^{-x+1} & \text{if } (x,y) \in V_1^- \end{cases}$$

such that

$$\left.\begin{aligned} t &= \frac{x+by-b-1}{x+by-b+b^2} \\ s &= \frac{bx+b^2y-b-b^2}{bx+b^2y-b+1} \\ b &= \tan \beta^* \end{aligned}\right\} \qquad (2.1.4)$$

and V_1, V_1^0, V_1^+, V_1^- are as illustrated earlier in Figure 2-1.

Proof:

The likelihood ratio function Λ is given by

$$\Lambda = \frac{e^{-x-y}}{\frac{1}{\hat{\hat{\theta}}_1\hat{\hat{\theta}}_2} e^{-x/\hat{\hat{\theta}}_1 - y/\hat{\hat{\theta}}_2}} \tag{2.1.5}$$

$$= \hat{\hat{\theta}}_1 \exp\left(-x\left[1-\frac{1}{\hat{\hat{\theta}}_1}\right]\right) \hat{\theta}_2 \exp\left(-y\left[1-\frac{1}{\hat{\hat{\theta}}_2}\right]\right)$$

where $(\hat{\hat{\theta}}_1, \hat{\hat{\theta}}_2)$ is the M.D.E. of (θ_1, θ_2) restricted to the cone V_1.

By using Lemma (2.1.1), we can easily get Λ for the cases of V_1, V_1^o, and V_1^-. For the case of V_1^+, we proceed as follows:

$$\Lambda = \left[\frac{x+by-b+b^2}{b^2+1}\right] \exp\left[-x\left(1-\frac{b^2+1}{x+by-b+b^2}\right)\right] \cdot \left[\frac{bx+b^2y-b+1}{b^2+1}\right]$$

$$\cdot \exp\left[-y\left(1-\frac{b^2+1}{bx+b^2y-b+1}\right)\right]$$

$$= \left[\frac{x+by-b+b^2}{b^2+1}\right] e^{-xt} \left[\frac{bx+b^2y-b+1}{b^2+1}\right] e^{-ys}$$

where t, s are given in (2.1.4). This ends the proof of Theorem (2.1.2).

Before deriving the L.R.T. we obtain the M.L.E. restricted to the cone V_1 with angle $\beta^* \in (0, \pi/2)$.

<u>**Lemma (2.1.3)**</u> The M.L.E. of (θ_1, θ_2) restricted to the cone V_1 is given by

$$(\hat{\theta}_1,\hat{\theta}_2)=\begin{cases}(x,y) & \text{if } (x,y)\in V_1\\ (1,1) & \text{if } (x,y)\in V_1^0\\ (d\ ,\ b(d-1)+1) & \text{if } (x,y)\in V_1^+\\ (x,1) & \text{if } (x,y)\in V_1^-\end{cases}$$

where $b = \tan\beta^*$ and d is a function of (x,y,b) which is a solution of the following cubic equation in θ_1 with $\theta_1 \geq 1$.

$$2b^2\theta_1{}^3+\theta_1{}^2[3b(1-b)-b^2x-by]+\theta_1[(1-b)^2-2b(1-b)x]-(1-b)^2x = 0$$

<u>Proof:</u>

Partition the sample space into 4 regions V_1, V_1^+, V_1^0 and V_1^- as illustrated in Figure 2-1.

Since the function

$$f(\theta_1,\theta_2)=\frac{1}{\theta_1\theta_2}\exp(-x/\theta_1-y/\theta_2)$$

has maximum point at $(\theta_1,\theta_2)=(x,y)$ and it is decreasing away from (x,y), then if $(x,y)\notin V_1$, the maximum point of $f(\theta_1,\theta_2)$ for $(\theta_1,\theta_2)\in V_1$ will be on the boundary of the cone V_1 .

Now, for $(x,y)\in V_1$, it is obvious that $(\hat{\theta}_1, \hat{\theta}_2)=(x,y)$.

Also , if $(x,y)\in V_1^-$ then $(\hat{\theta}_1, \hat{\theta}_2) = (x,1)$. It is left to obtain the M.L.E. for the cases $(x,y)\in V_1{}^+$ and $(x,y)\in V_1{}^0$. For the case that $(x,y)\in V_1{}^+$, the M.L.E. is located at the upper boundary of V_1, so

$$\theta_2 = b\ (\theta_1 - 1) + 1$$

Thus, we want to find the maximum value of the function

$$g(\theta_1) = \frac{1}{\theta_1[b(\theta_1-1)+1]} \exp\left(-x/\theta_1 - y/ [b(\theta_1-1)+1] \right)$$

for $\theta_1>1$. But,

$$\ln g(\theta_1) = -\ln \theta_1 - x/\theta_1 - \ln (b(\theta_1-1)+1) - y/ [b(\theta_1-1)+1]$$

So that,

$$\frac{\partial}{\partial\theta_1} \ln g(\theta_1) = -\frac{1}{\theta_1} - \frac{b}{b(\theta_1-1)+1} + \frac{x}{\theta_1^2} + \frac{b\ y}{[b(\theta_1-1)+1]^2} = 0$$

This gives

$$2b^2\theta_1^3 + b(1-b)\theta_1^2 + 2b\theta_1^2(1-b) + \theta_1(1-b)^2 = b^2\theta_1^2x + (1-b)^2x + 2b\theta_1(1-b)x + by\theta_1^2$$

which is equivalent to

$$h(\theta_1) = 0, \qquad (2.1.6)$$

where,

$$h(\theta_1)=2b^2\theta_1{}^3+\theta_1{}^2[3b(1-b)-b^2x-by]+\theta_1[(1-b)^2-2b(1-b)x]-(1-b)^2x$$

At $\theta_1=1$,we get

$$h(1)=b(1-y)+(1-x).$$

Since we are treating the case $(x,y)\in V_1{}^+$, then

$$b(1-y) < -(1-x) ,$$

therefore, $h(1) < 0$.

Also

$$\lim_{\theta_1\to\infty} (h(\theta_1)/\theta_1^3) = 2b^2$$

which is greater than 0 . This implies that $h(\infty) > 0$. So, we can say that $h(\theta_1)$ takes both negative and positive values on the range $(1, \infty)$. Also, since $h(\theta_1)$ is continuous on this range, it must has a root which is greater than 1 .

Let the root of equation (2.1.6) be denoted by "d",

i.e., $\hat{\theta}_1 \equiv d$, say

Therefore $\hat{\theta}_2 = b(d-1)+1$.

For $(x,y) \in V_1^o$, we have two cases that either $(\hat{\theta}_1, \hat{\theta}_2)$ is at the lower boundary of the cone V_1 i.e , $\theta_2 = 1$ or $(\hat{\theta}_1, \hat{\theta}_2)$ is at the upper boundary of the cone V_1 i.e,

$$\theta_2 = b(\theta_1-1)+1 .$$

For the first case we have to minimize

$$f(\theta_1)=(1/\theta_1)\exp(-x/\theta_1-y).$$

But
$$\ln f(\theta_1) = -\ln(\theta_1) - x/\theta_1 - y$$

$$\frac{\partial \ln f(\theta_1)}{\partial \theta_1} = \frac{1}{\theta_1}\left(\frac{x}{\theta_1} - 1\right)$$

Since $x<1$ and $\theta_1 > 1$, then $\frac{x}{\theta_1} < 1$, therefore, $\frac{\partial \ln f(\theta_1)}{\partial \theta_1} < 0$. This implies that $f(\theta_1)$ is a decreasing function for $\theta_1 > 1$. Thus its maximum will be at $\theta_1=1$. Therefore the M.L.E.

$$(\hat{\theta}_1, \hat{\theta}_2) = (1, 1) .$$

In the second case the M.L.E. will be a solution of the cubic equation given by (2.1.6). In this case, we have to show that this cubic equation has no roots for $\theta_1 > 1$, which is equivalent to showing that

$$h(\theta_1) > 0 \quad \text{for all } \theta_1 > 1,$$

where $h(\theta_1)$ is given by (2.1.6). To show this, it is enough to show that

$$\min_{(x,y)\in V_1^o} h(\theta_1) > 0 \quad \text{for } \theta_1 > 1$$

It can be seen that the minimum is at $(x,y)=(1,1)$, Therefore, it is enough to show that the function $h_1(\theta_1) > 0$ for $\theta_1>1$, where h_1 is given by

$$h_1(\theta_1) = h(\theta_1)\Big|_{x=1,y=1} = 2b^2\theta_1^3+\theta^2[4b-2b-4b^2] + (1-b)(1-3b)\theta -(1-b)^2.$$

But $h_1(\theta_1)$ can be written as follows,

$$h_1(\theta_1) = b^2(2\theta_1{}^3-4\theta_1{}^2+2\theta_1-1)+b(2\theta_1{}^2-4\theta_1+2)+\theta_1-1,$$

which can be written as,

$$h_1(\theta_1) = [\theta_1-1]\ [b^2(\theta_1{}^2+(\theta_1-1)^2) + 2b(\theta_1-1) + 1].$$

Notice that for $\theta_1>1$, $h_1(\theta_1) > 0$. This implies that the function $g(\theta_1)$, is decreasing for $\theta_1>1$ and its maximum will be at $\theta_1= 1$ and so $\theta_2 = 1$, therefore we get that the M.L.E. $(\hat{\theta}_1, \hat{\theta}_2) = (1, 1)$. This ends the proof.

Now, we derive the L.R.T.

Theorem (2.1.4)

Consider the bivariate independent exponential distribution with parameters (θ_1,θ_2). For the testing problem (2.1.1) with angle $\beta^*\in(0, \pi/2)$ the L.R.T. ϕ'_{V_1} is of the form

$$\phi'_{V_1} = \begin{cases} 0 & \Lambda' \geq k'_1 \\ 1 & \Lambda' < k'_1 \end{cases} \qquad (2.1.7)$$

where k'_1 is a constant based on the level of significance, and

$$\Lambda' = \begin{cases} x\, y\, e^{-x-y+2} & \text{if } (x,y) \in V_1 \\ 1 & \text{if } (x,y) \in V_1^0 \\ Q(x,y) & \text{if } (x,y) \in V_1^+ \\ x\, e^{-x+1} & \text{if } (x,y) \in V_1^- \end{cases}$$

where

$$Q(x,y) = d \exp\left[-x \left(\frac{d-1}{d}\right)\right] \left[b(d-1)+1 \right] \exp\left[-y \left(\frac{b(d-1)}{b(d-1)+1}\right)\right]$$

and d is given in Lemma (2.1.3).

Proof:

The likelihood ratio statistics Λ' is given by

$$\Lambda' = \hat{\theta}_1 \exp\left[-x\left[1-\frac{1}{\hat{\theta}_1}\right]\right] \hat{\theta}_2 \exp\left(-y\left[1-\frac{1}{\hat{\theta}_2}\right]\right)$$

where $(\hat{\theta}_1, \hat{\theta}_2)$ is the M.L.E. of (θ_1, θ_2) in the cone V_1. By using Lemma (2.1.3), we can easily get Λ' for the cases V_1, V_1^0 and V_1^-. For the case V_1^+, we have

$$\Lambda' = d \exp\left[-x \left(\frac{d-1}{d}\right)\right] \left[b(d-1)+1 \right] \exp\left[-y \left(\frac{b(d-1)}{b(d-1)+1}\right)\right]$$

$$= Q(x,y) .$$

This ends the proof.

There is one more test that will be considered in this chapter which is of the L.R.T. form except for its value in the cone V_1^+, we will call such test the Modified L.R.T., it is given by

$$\phi V_1'' = \begin{cases} 0 & \Lambda'' \geq k_1'' \\ 1 & \Lambda'' < k_1'' \end{cases} \qquad (2.1.8)$$

where k_1'' is a constant based on the level of significance, and

$$\Lambda'' = \begin{cases} x\ y\ e^{-x-y+2} & \text{if } (x,y) \in V_1 \\ 1 & \text{if } (x,y) \in V_1^o \\ y + \frac{x}{b} + C_o & \text{if } (x,y) \in V_1^+ \\ x\ e^{-x+1} & \text{if } (x,y) \in V_1^- \end{cases}$$

where C_o is chosen so that the line $y+x/b+C_o = k_1''$ intersect the curve $x\ y\ e^{-x-y+2}$ at the point (x_o, y_o) which is the intersection between the curve and the line $y = b(x-1)+1$

Now, we show that the M.L.E. coincide with M.D.E. for $\beta^* = \pi / 4$, and $\pi / 2$.

Lemma (2.1.5) For $\beta^* = \pi/4$ and $\pi/2$, the M.L.E. of (θ_1, θ_2) coincide with M.D.E. of (θ_1, θ_2).

Proof:

(1) For $\beta^* = \pi / 4$, the M.D.E. of (θ_1, θ_2) given by

$$(\hat{\hat{\theta}}_1, \hat{\hat{\theta}}_2) = \begin{cases} (x,y) & \text{if } (x,y) \in V_1 \\ (1,1) & \text{if } (x,y) \in V_1^o \\ (\frac{x+y}{2}, \frac{x+y}{2}) & \text{if } (x,y) \in V_1^+ \\ (x,1) & \text{if } (x,y) \in V_1^- \end{cases} \qquad (2.1.9)$$

Also, the M.L.E. of (θ_1,θ_2) as given by (2.1.9), where $\hat{\theta}_1=\frac{x+y}{2}$ is a solution of the following cubic equation:

$$\theta_1^3 + \theta_1^2\,[-x/2 - y/2] = 0$$

(2) For $\beta^* = \pi / 2$, the M.D.E. of (θ_1,θ_2) given by

$$(\hat{\hat{\theta}}_1,\hat{\hat{\theta}}_2)=\begin{cases}(x,y) & \text{if } (x,y) \in V_1\\ (1,1) & \text{if } (x,y) \in V_1^0\\ (1,y) & \text{if } (x,y) \in V_1^+\\ (x,1) & \text{if } (x,y) \in V_1^-\end{cases} \qquad (2.1.10)$$

Also, the M.L.E. of (θ_1,θ_2) as defined by (2.1.10), where $(\hat{\theta}_1,\hat{\theta}_2) = (1,y)$ is the maximum point when $x < 1$, $y > 1$.

Using Lemma (2.1.5), we show that, for the cases $\beta^*=\pi/4$, and $\beta^*=\pi/2$, the three tests are equivalent.

Theorem (2.1.6)

The three tests ϕ_{V_1}, ϕ'_{V_1}, ϕ''_{V_1} coincide for the case $\beta^* = \pi / 4$ and $\pi / 2$.

Proof: We observe that, the only difference is for the values in the cone V_1^+. For the case $\beta^* = \pi / 4$. It can be seen that for the test ϕ_{V_1}, if $(x,y) \in V_1^+$, we have

$$\Lambda = \left(\frac{x+y}{2}\right)^2 \exp(-x-y+2) \quad < \quad k$$

which is equivalent to $x + y < k^*$.

Also, for the test ϕ'_{V_1}, if $(x,y) \in V_1^+$, we have

$$\Lambda' = \left(\frac{x+y}{2}\right)^2 \exp(-x-y+2) < k$$

which is equivalent to $x+y < k^*$.

Furthermore for $(x,y) \in V_1^+$, we have $\Lambda'' = x + y < k^*$, which is equivalent to ϕ_{V_1} and ϕ'_{V_1}.

Now, for the case $\beta^* = \pi / 2$, for the test ϕ_{V_1}, if $(x,y) \in V_1^+$ we get

$$\Lambda = y \exp(-y + 1) < k$$

which is equivalent to $y < C^*$.

For the test ϕ'_{V_1}, if $(x,y) \in V_1^+$, we get

$$\Lambda' = y \exp(-y + 1) < k$$

which is equivalent to $y < C^*$.

And, for the test ϕ''_{V_1}, if $(x,y) \in V_1^+$ then

$$\Lambda'' = y < C^*$$

which is equivalent to ϕ_{V_1}, ϕ'_{V_1}.

Theorem (2.1.7)

Consider the bivariate independent exponential distribution with parameters (θ_1,θ_2). For the testing problem (2.1.1) with cone V_2, the whole space, the L.R.T. ϕ_{V_2} has the form

$$\phi_{V_2} = \begin{cases} 0 & \Lambda^* \geq k_2 \\ 1 & \Lambda^* < k_2 \end{cases}$$

where

$$\Lambda^* = x\, y\, e^{-x-y+2}$$

and k_2 is a constant determined by the level of significance.

Proof.

The likelihood ratio function Λ^* is given by

$$\Lambda^* = \frac{e^{-x-y}}{\sup\limits_{(\theta_1,\theta_2)\in V_2} \frac{1}{\theta_1\theta_2} e^{-x/\theta_1 - y/\theta_2}}$$

$$= \hat{\theta}_1 \exp\left(-x\left[1-\frac{1}{\hat{\theta}_1}\right]\right)\hat{\theta}_2 \exp\left(-y\left[1-\frac{1}{\hat{\theta}_2}\right]\right)$$

where $(\hat{\theta}_1,\hat{\theta}_2)$ is the M.L.E. of (θ_1,θ_2) corresponding to the positive quadrant space.

It is obvious that the M.L.E. of (θ_1,θ_2) will be (x,y), this implies that

$$\Lambda^* = x\ y\ e^{-x-y+2}$$

This ends of the proof.

Notice that the M.D.L.R.T. is the same as the L.R.T. for unrestricted alternative.

2.2 Derivation of the Power Functions

In this section, the power functions for the L.R.T., M.D.L.R.T. and the Modified L.R.T corresponding to the cones V_1 and V_2 are derived.

Theorem (2.2.1)

Consider the Modified L.R.T. given by (2.1.8). For any alternative $(\theta_1,\theta_2) \in V_1$, let (Δ,β) be the transformation of

(θ_1,θ_2) given by

$$A = \theta_1\ \theta_2\ e^{-\theta_1-\theta_2+2}$$

and $$\beta = \text{arc tan}\left[\frac{\theta_2-1}{\theta_1-1}\right]$$

Then the power function of the Modified L.R.T. as a function of (A,β) is given by

$$\mathscr{B}_1(A,\beta) = H_1^*(b,\theta_1) + e^{-A/\theta_1-[b(A-1)+1]/\theta_2} + \frac{b\theta_2}{b\theta_2-\theta_1}\ e^{-C/\theta_2}$$

$$\cdot\left[1-e^{-A/\theta_1+A/(b\theta_2)}\right] + e^{-C_4/\theta_1}\left(1 - e^{-1/\theta_2}\right) \quad (2.2.1)$$

where $$H_1^*(b,\theta_1) = \frac{1}{\theta_2}\int_1^{b(A-1)+1} e^{-(y/\theta_2)-(w_1(y)/\theta_1)}dy ,$$

$A = b(b+c-1) / (b^2+1)$ is the intersection point between $y=c-(x/b)$ and $y = b(x-1)+1$, C_4 is solution of $C_4e^{-C_4}=k_1^{\prime\prime}e^{-1}$, such that $C_4 > 1$, $\theta_2 = (\theta_1-1)\tan\beta + 1;\ 0 \le \beta \le \beta^*$ and $w_1(y) = f^{-1}\ [(k_1^{\prime\prime}/y)e^{y-2}]$ such that $f(x) = xe^{-x}$

<u>Proof:</u>

For fixed A, θ_1,θ_2 are related as

$$\left.\begin{array}{l} \theta_1 \, e^{-\theta_1} = \dfrac{\Delta}{\theta_2} \, e^{\theta_2 - 2} \\ \text{and} \\ \beta = \arc\tan\left[\dfrac{\theta_2 - 1}{\theta_1 - 1}\right] \end{array}\right\} \qquad (2.2.2)$$

From (2.2.2) we have

$$\theta_1 \, e^{-\theta_1} = \frac{\Delta}{1+(\theta_1-1)\tan\beta} \, e^{(\theta_1-1)\tan\beta - 1}$$

this implies that:

$$\theta_1 + \theta_1(\theta_1-1)\tan\beta = \Delta \, e^{(\theta_1-1)(1+\tan\beta)} \qquad (2.2.3)$$

i.e., $\theta_1 \equiv$ function of $(\Delta,\beta) \equiv G(\beta,\Delta)$, say.

We can represent (2.2.2) graphically as follows:

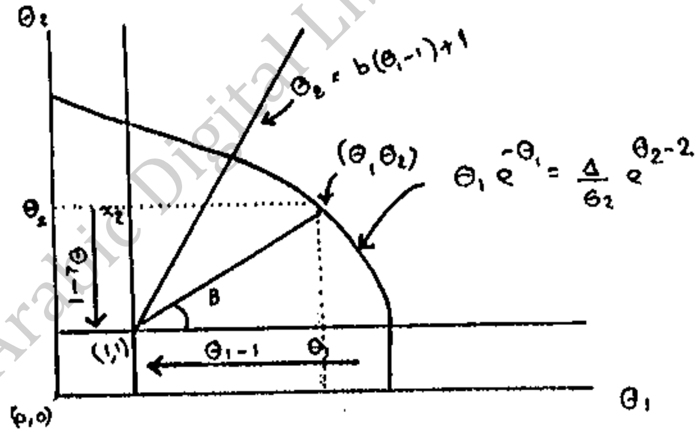

Figure 2-2

But the power function in terms of θ_1, θ_2 is given by

$$E_{\theta_1,\theta_2} \, \phi''_{V_1} = 1 - \sum_{i=1}^{6} P_{\theta_1,\theta_2}(B_i)$$

where B_i; $i=1,\ldots,6$ are shown in the graph below, and $P_{\theta_1,\theta_2}(.)$ is the pf under the density (1.1.1).

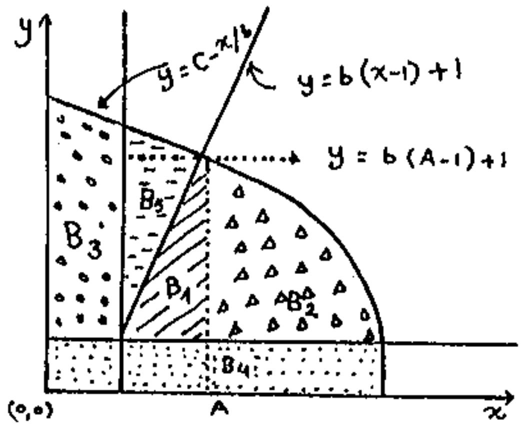

Figure 2-3

Sketch of the acceptance region of the Modified L.R.T. ϕ''_{V_1}

Now ,

$$P_{\theta_1,\theta_2}(B_1) = \int_1^A \int_1^{b(x-1)+1} \frac{1}{\theta_1} e^{-x/\theta_1} \frac{1}{\theta_2} e^{-y/\theta_2}\, dy\, dx$$

$$= e^{-1/\theta_1-1/\theta_2} - e^{-A/\theta_1-1/\theta_2} + \frac{\theta_2}{\theta_2+b\theta_1}$$

$$\cdot\, e^{-A/\theta_1-[b(A-1)+1]/\theta_2} - \frac{\theta_2}{\theta_2+b\theta_1} e^{-1/\theta_1-1/\theta_2}$$

Also,

$$P_{\theta_1,\theta_2}(B_2) = \int_1^{b(A-1)+1} \int_A^{w_1(y)} \frac{1}{\theta_1} e^{-x/\theta_1} \frac{1}{\theta_2} e^{-y/\theta_2}\, dx\, dy$$

$$= e^{-A/\theta_1-1/\theta_2} - e^{-A/\theta_1-[b(A-1)+1]/\theta_2} - H_1^*(b,\theta_1)$$

where $H_1^*(b,\theta_1) = \frac{1}{\theta_2}\int_1^{b(A-1)+1} e^{-y/\theta_2-w_1(y)/\theta_1}\, dy$

$$P_{\theta_1,\theta_2}(B_3) = \int_0^1 \int_1^{C-x/b} \frac{1}{\theta_1} e^{-x/\theta_1} \frac{1}{\theta_2} e^{-y/\theta_2} \, dy \, dx$$

$$= e^{-1/\theta_2}\left[1-e^{-1/\theta_1}\right] + \frac{b\theta_2}{b\theta_2-\theta_1} e^{-C/\theta_2}\left[e^{-1/\theta_1+1/(b\theta_2)}-1\right]$$

$$P_{\theta_1,\theta_2}(B_4) = \int_0^{C_4} \int_0^1 \frac{1}{\theta_1} e^{-x/\theta_1} \frac{1}{\theta_2} e^{-y/\theta_2} \, dy \, dx$$

$$= 1 - e^{-C_4/\theta_1} - e^{-1/\theta_2} + e^{-C_4/\theta_1-1/\theta_2}$$

and

$$P_{\theta_1,\theta_2}(B_5) = \int_1^A \int_{b(x-1)+1}^{C-(x/b)} \frac{1}{\theta_1} e^{-x/\theta_1} \frac{1}{\theta_2} e^{-y/\theta_2} \, dy \, dx$$

$$= -\frac{\theta_2}{\theta_2+b\theta_1}\left[e^{-A/\theta_1-[b(A-1)+1]/\theta_2} - e^{-1/\theta_1-1/\theta_2}\right]$$

$$+ \frac{b\theta_2}{b\theta_2-\theta_1} e^{-C/\theta_2}\left[e^{-A/\theta_1+A/(b\theta_2)} - e^{-1/\theta_1+1/(b\theta_2)}\right]$$

This implies that the power function is given by (2.2.1).

This ends the proof.

For the special case that $\beta^*=\pi/2$, it is easily seen that: $\frac{b\theta_2}{b\theta_2-\theta_1}$, $b(A-1)+1$, A/b, and A tend to 1, C, 0, and 1, respectively, as b goes to infinity. Also, the value of C will be equal to C_4. Therefore the power function reduces to

$$\mathcal{B}_1(\Lambda,\beta) = \frac{1}{(\theta_1-1)\tan\beta+1}\int_1^{C_4} e^{-y/[(\theta_1-1)\tan\beta+1]-w_1(y)/\theta_1}\, dy$$

$$+e^{-C_4/[(\theta_1-1)\tan\beta+1]} + e^{-C_4/\theta_1}\left[1-e^{-1/[(\theta_1-1)\tan\beta+1]}\right] \quad (2.2.4)$$

Theorem (2.2.3)

Consider the L.R.T. given by Theorem (2.1.4). For any $(\theta_1,\theta_2) \in V_1$. Let (Λ,β) be the transformation of (θ_1,θ_2) given by (2.2.2). Then the power function of the L.R.T. in terms of (Λ,β) is given by

$$\bar{\mathcal{B}}(\Lambda,\beta) = e^{-C_4/\theta_1} + e^{-1/\theta_2} - e^{-C_4/\theta_1-1/\theta_2}$$

$$- \int_1^{G_1}\int_1^{b(x-1)+1} \frac{1}{\theta_1} e^{-x/\theta_1} \frac{1}{\theta_2} e^{-y/\theta_2}\, dy\, dx$$

$$- \int_1^{G_2}\int_0^{b(x-1)+1} \frac{1}{\theta_1} e^{-x/\theta_1} \frac{1}{\theta_2} e^{-y/\theta_2}\, dx\, dy \quad (2.2.5)$$

where G_1, G_2 are two curves defined by $x\, e^{-x} = (k_1'/y)\; e^{y-2}$ and $Q(x,y) = k_1'$, respectively.

Proof:

$$\text{Power} \equiv E_{(\theta_1,\theta_2)}\, \phi_{V_1}' = 1 - \left[p(A_1) + p(A_2) + p(A_3)\right]$$

where A_i's $(i = 1, 2, 3)$ are shown in figure 2-4.

Now,

$$p(A_1) = \int_0^{C_4}\int_0^{1} \frac{1}{\theta_1} e^{-x/\theta_1} \frac{1}{\theta_2} e^{-y/\theta_2}\, dy\, dx$$

$$= 1 - e^{-C_4/\theta_1} - e^{-1/\theta_2} + e^{-C_4/\theta_1-1/\theta_2}$$

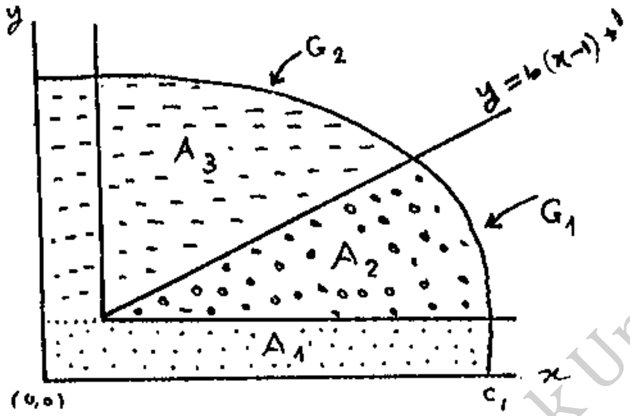

Figure 2-4

Sketch of the acceptance region of the L.R.T. ϕ'_{V_1}.

$$p(A_2) = \int_1^{G_1} \int_1^{b(x-1)+1} \frac{1}{\theta_1} e^{-x/\theta_1} \frac{1}{\theta_2} e^{-y/\theta_2} \, dy \, dx$$

and
$$p(A_3) = \int_1^{G_2} \int_0^{b(x-1)+1} \frac{1}{\theta_1} e^{-x/\theta_1} \frac{1}{\theta_2} e^{-y/\theta_2} \, dx \, dy$$

This ends the proof.

Notice that the power function of M.D.L.R.T. has the same form as (2.2.5) except G_2 is the curve defined by

$$\left[\frac{x+by-b+b^2}{b^2+1}\right] e^{-xt} \left[\frac{bx+b^2y-b+1}{b^2+1}\right] e^{-ys} = k_1$$

where b, t and s are defined by (2.1.4).

Theorem (2.2.4)

Consider the L.R.T. given by Theorem (2.1.6). For any $(\theta_1,\theta_2)\in V_2$, the power function is given by:

$$\beta_2(\theta_1,\theta_2) = 1 + e^{-1/\theta_1 - C_1/\theta_2} + \frac{1}{\theta_2}\int_1^{C_1} e^{-y/\theta_2 - w_2(y)/\theta_1}\,dy$$

$$- e^{-1/\theta_2-(2-C_1)^+/\theta_1} + \frac{1}{\theta_1}\int_{(2-C_1)^+}^{1} e^{-x/\theta_1 - w_3(x)/\theta_2}\,dx$$

$$-\frac{1}{\theta_2}\int_{(2-C_1)^+}^{1} e^{-y/\theta_2-(2-\bar{w}_2(y))^+/\theta_1} - e^{-y/\theta_2-\bar{w}_2(y)/\theta_1}\,dy \qquad (2.2.6)$$

where

$$w_2(y)=f^{-1}\left[\frac{k_2}{y}\,e^{y-2}\right],\quad w_3(x)=f^{-1}\left[\frac{k_2}{2-x}\,e^{-x}\right],\quad \bar{w}_2(y)=f^{-1}\left[\frac{k_2}{2-y}\,e^{-y}\right]$$

such that $f(x) = x\,e^{-x}$ and $H^+ = \max(H,0)$

<u>Proof:</u>

$$\text{Power} \equiv E_{(\theta_1,\theta_2)}\,\phi_{V_2} = 1 - \sum_{i=1}^{4} P_{(\theta_1,\theta_2)}(R_i)$$

where R_i; $i = 1,2,3,4$ are as in the following sketch:

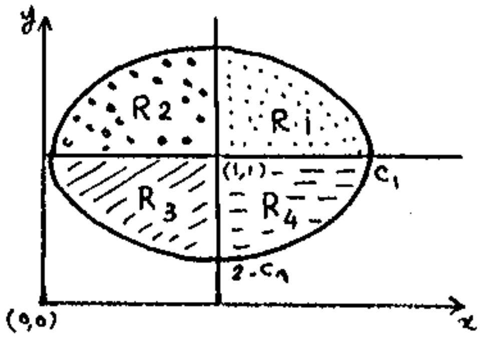

Figure 2-5

(when $C_1 \le 2$)

Sketch of the acceptance region of the LRT. ϕ_{V_2}

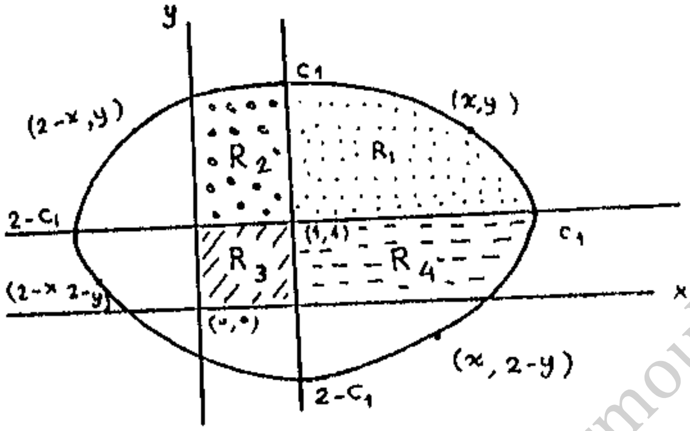

Figure 2-6

(When $C_1 > 2$)

Sketch of the acceptance region of the LRT. ϕ_{V_2}

Notice that,

$$\text{under } R_1 \ , \ x = f^{-1}\left[\frac{k_2}{y}\, e^{y-2}\right] \equiv w_2(y) \ , \ \text{say}$$

$$\text{under } R_2 \ , \ 2-x = f^{-1}\left[\frac{k_2}{y}\, e^{y-2}\right] \Rightarrow x = 2 - w_2(y)$$

$$\text{under } R_3 \ , \ 2-x = f^{-1}\left[\frac{k_2}{2-y}\, e^{-y}\right] \Rightarrow x = 2 - f^{-1}\left[\frac{k_2}{2-y}\, e^{-y}\right]$$

and $$\text{under } R_4 \ , \ x = f^{-1}\left[\frac{k_2}{2-y}\, e^{-y}\right] \equiv \bar{w}_2(y)$$

where $$f(Z) = Z\, e^{-Z}$$

Now,

$$P_{(\theta_1,\theta_2)}(R_1) = \int_1^{C_1}\int_1^{w_2(y)} \frac{1}{\theta_1}\, e^{-x/\theta_1}\, \frac{1}{\theta_2}\, e^{-y/\theta_2}\, dx\, dy$$

$$=e^{-1/\theta_1}\left[e^{-1/\theta_2}-e^{-C_1/\theta_2}\right]-\frac{1}{\theta_2}\int_1^{C_1} e^{-y/\theta_2-w_2(y)/\theta_1}\,dy$$

Also,

$$P_{(\theta_1,\theta_2)}(R_2) = \int_{(2-C_1)^+}^{1}\int_1^{w_3(x)} \frac{1}{\theta_1} e^{-x/\theta_1}\frac{1}{\theta_2} e^{-y/\theta_2}\,dy\,dx$$

$$=\int_{(2-C_1)^+}^{1} \frac{1}{\theta_1} e^{-x/\theta_1-1/\theta_2}-\frac{1}{\theta_1} e^{-x/\theta_1-w_3(x)/\theta_2}\,dx$$

$$= e^{-1/\theta_2}\left(e^{-(2-C_1)^+/\theta_1}-e^{-1/\theta_1}\right)$$

$$-\frac{1}{\theta_1}\int_{(2-C_1)^+}^{1} e^{-x/\theta_1-w_3(x)/\theta_2}\,dx$$

$$= e^{-1/\theta_2-(2-C_1)^+/\theta_1}-e^{-1/\theta_1-1/\theta_2}$$

$$-\frac{1}{\theta_1}\int_{(2-C_1)^+}^{1} e^{-x/\theta_1-w_3(x)/\theta_2}\,dx$$

$$P_{(\theta_1,\theta_2)}(R_3) = \int_{(2-C_1)^+}^{1}\int_{(2-\bar{w}_2(y))^+}^{1} \frac{1}{\theta_1} e^{-x/\theta_1}\frac{1}{\theta_2} e^{-y/\theta_2}\,dx\,dy$$

$$= e^{-1/\theta_1-1/\theta_2}-e^{-1/\theta_1-(2-C_1)^+/\theta_2}$$

$$+\frac{1}{\theta_2}\int_{(2-C_1)^+}^{1} e^{-y/\theta_2-(2-\bar{w}_2(y))^+/\theta_1}\,dy$$

and

$$P_{(\theta_1,\theta_2)}(R_4) = \int_{(2-C_1)^+}^{1} \int_{1}^{\bar{w}_2(y)} \frac{1}{\theta_1} e^{-x/\theta_1} \frac{1}{\theta_2} e^{-y/\theta_2}\, dx\, dy$$

$$= e^{-1/\theta_1-(2-C_1)^+/\theta_2} - e^{-1/\theta_1-1/\theta_2}$$

$$- \frac{1}{\theta_2} \int_{(2-C_1)^+}^{1} e^{-y/\theta_2-\bar{w}_2(y)/\theta_1}\, dy$$

So, the power function in terms of (θ_1,θ_2) is given by:

$$\beta_2(\theta_1,\theta_2) = 1 - e^{-1/\theta_1-(2-C_1)^+/\theta_1} + e^{-1/\theta_1-C_1/\theta_2}$$

$$+ \frac{1}{\theta_2} \int_{1}^{C_1} e^{-y/\theta_2-w_2(y)/\theta_1}\, dy + \frac{1}{\theta_1} \int_{(2-C_1)^+}^{1} e^{-x/\theta_1-w_3(x)/\theta_2}\, dx$$

$$+ \frac{1}{\theta_2} \int_{(2-C\ )^+}^{1} e^{-y/\theta_2-\bar{w}_2(y)/\theta_1} - e^{-y/\theta_2-(2-w_2(y))^+/\theta_1}\, dy$$

The power function given by (2.2.6) can be written in other forms as follows:

i. If the $C_1 \le 2$ then the power function becomes:

$$\beta_2(\theta_1,\theta_2)=1-e^{-1/\theta_2-(2-C_1)/\theta_1}+e^{-1/\theta_1-C_1/\theta_2}$$

$$+\frac{1}{\theta_2} \int_{1}^{C_1} e^{-y/\theta_2-w_2(y)/\theta_1}\, dy + \frac{1}{\theta_1} \int_{(2-C_1)}^{1} e^{-x/\theta_1-w_3(x)/\theta_2}\, dx$$

$$+ \frac{1}{\theta_2} \int_{(2-C_1)}^{1} e^{-y/\theta_2-\bar{w}_2(y)/\theta_1} - e^{-y/\theta_2-(2-\bar{w}_2(y))/\theta_1}\, dy$$

ii. If $C_1 > 2$ then the power function is given by:

$$\beta_2(\theta_1,\theta_2) = 1 - e^{-1/\theta_2} - \frac{1}{\theta_2}\int_0^1 e^{-y/\theta_2}\,dy + e^{-1/\theta_1 - C_1/\theta_2}$$

$$+ \frac{1}{\theta_2}\int_1^{C_1} e^{-y/\theta_2 - w_2(y)/\theta_1}\,dy + \frac{1}{\theta_1}\int_0^1 e^{-x/\theta_1 - w_3(x)/\theta_2}\,dx$$

$$+ \frac{1}{\theta_2}\int_0^1 e^{-y/\theta_2 - \bar{w}_2(y)/\theta_1}\,dy$$

$$\beta_2(\theta_1,\theta_2) = e^{-1/\theta_1 - C_1/\theta_2} + \frac{1}{\theta_2}\int_1^{C_1} e^{-y/\theta_2 - w_2(y)/\theta_1}\,dy$$

$$+ \frac{1}{\theta_1}\int_0^1 e^{-x/\theta_1 - w_3(x)/\theta_2}\,dx + \frac{1}{\theta_2}\int_0^1 e^{-y/\theta_2 - \bar{w}_2(y)/\theta_1}\,dy$$

This ends the proof.

CHAPTER THREE

SOME PROPERTIES OF THE

M.D.L.R.T., L.R.T. AND MODIFIED L.R.T.

3.1 Introduction

This chapter gives some properties of the L.R.T., M.D.L.R.T. and Modified L.R.T. For certain values of β^*, it is shown that some of these tests are admissible tests in the case of zero-one loss function. In addition, through some computations, it is observed that for fixed Δ, the power functions of the L.R.T., M.D.L.R.T. and Modified L.R.T. are monotone in β for $\beta<\bar{\beta}$ and symmetric about $\bar{\beta}$, where $\bar{\beta}$ is determined by the angle β^* of the cone. Furthermore, we prove that for the case $\beta^* = \pi/2$, the power function has maximum value at $\pi/4$ and two equal minima at 0 and $\pi/2$.

We will mostly concentrate on the Modified L.R.T., for many reasons: its simplicity to deal with and its closeness to the other two tests. In addition, for some β^*, it is equivalent to the other two tests. Also it has some optimal properties.

3.2 Admissibility

In this section, we will show that for particular values of β^*, the L.R.T., M.D.L.R.T. and Modified L.R.T. for the testing problem given by (1.1.2) have a V-decreasing and convex acceptance region, therefore by Eaton's (1970) Theorem, these tests are admissible tests for the given case.

Theorem (3.2.1)

Consider the statistical model (1.1.1) and the testing problem (1.1.3). For $\beta^* = 2\pi$ (which is the unrestricted case) the L.R.T. given by Lemma (2.1.7) is an admissible test.

Proof:

Let A be the acceptance region of L.R.T. A can be written as $A = \{Q(x)\ Q(y) \geq K_2^*\}$ where $Q(t) = t\ e^{-t}$ and K_2^* is some constant.

To show that A is convex, it is enough to show that the curve given by

$$Q(x)\ Q(y) = k_2^*$$

is concave down function for $y > 1$ and concave up for $y < 1$.

We have

$$\frac{\partial y}{\partial x} = \frac{y(x-1)}{x(1-y)}$$

and

$$\frac{\partial^2 y}{\partial x^2} = \frac{y}{(1-y)^3 x^2}\left[2x^2y+2x-x^2+y^2-4yx\right]$$

$$= \frac{y}{(1-y)^3 x^2}\left[2x(x-1)(y-1)+(x-y)^2\right]$$

For $y > 1$, we have the following function

$$f(x) = x^2(2y-1) + x(2-4y) + y^2$$

also, $f'(x) = 2x(2y-1) + (2-4y)$

Let $a = 2y-1,\ b = -2a$ and $c = y^2$

So, $x = (4y-2) / (4y-2) = 1$,

$$b^2 - 4ac = -4a(y-1)^2 < 0$$

Then f(x) has no real root, therefore

$$f(x) > 0 \quad \text{for all } x.$$

Similary, for $y < 1$, we get

$$\frac{\partial^2 y}{\partial x^2} > 0$$

This shows that A is convex. Birnbaum (1955) theorem implies that the L.R.T., for unrestricted case is an admissible test. This ends the proof.

For restricted case with $\beta^* \le \pi/2$ the form of the L.R.T. has a curve in V_1^+ which is not easy to show that its concave down, especially we have to solve a cubic equation and we use its solution to get Q(x,y) which is defined in Theorem (2.1.4). The difficulty is to find the derivative $\partial^2 y/\partial x^2$ and to show that it is negative, therefore the Modified L.R.T. is used in most cases.

For the Modified L.R.T. we show that it is admissible for the case that $\beta^* \ge \pi/4$ and inadmissible for $\beta^* < \pi/4$.

Lemma (3.2.2)

Consider the statistical model (1.1.1) and the testing problem (1.1.3). The Modified L.R.T. has a convex acceptance region for the cases $\pi/4 \le \beta^* \le \pi/2$.

Proof:

It is enough to show the following:

(1) The curve G_1 is concave down function.

(2) The slope of the tangent line L_1 of the curve G_1 at (x_0, y_0) is less than or equal to the slope of the line L_2, where G_1, L_1 and L_2 are illustrated in the following figure.

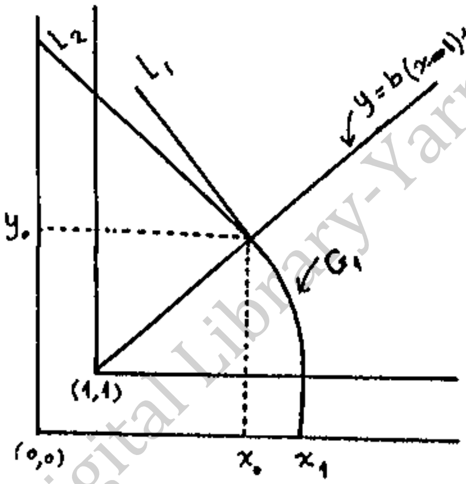

Figure 3-1

Illustration of the curve G_1 and the tangent lines L_1 and L_2.

The curve G_1 has the form:

$$x\, e^{-x}\, y\, e^{-y} = K_1''\, e^{-2}$$

The equation of the line L_1 is

$$(y-y_0) = m_1\,(x-x_0)$$

and the equation of the line L_2 is

$$y = m_2\,(x-x_0) + y_0$$

where $m_1 = \left.\dfrac{\partial y}{\partial x}\right|_{(x_0, y_0)}$ and $m_2 = -1 / b$

We want to show that the curve G_1 is concave down or $\frac{\partial^2 y}{\partial x^2} < 0$

But

$$\frac{\partial y}{\partial x} = \frac{xy-y}{x-yx}$$

and

$$\frac{\partial^2 y}{\partial x^2} = \frac{y}{(1-y)^3 x^2}\left[x(x-1)(y-1)+x(1-y)^2-(x-1)(y-1)-(1-y)^2(x-1)+(x-1)^2 y\right]$$

$$= \frac{y}{(1-y)^3 x^2}\left[(x-1)(y-1)(x-y)+x-2yx+y^2x+yx^2+y-2yx\right]$$

$$= \frac{y}{(1-y)^3 x^2}\left[2x^2y+2x-x^2+y^2-2yx-2yx\right]$$

$$= \frac{y}{(1-y)^3 x^2}\left[2xy(x-1)-2x(x-1)+(x^2+y^2-2yx)\right]$$

$$= \frac{y}{(1-y)^3 x^2}\left[2x(x-1)(y-1)+(x-y)^2\right]$$

Since $x>1$, $y>1$, we get $\partial^2 y/\partial x^2 < 0$. This proves that the curve G_1 is concave down.

Now, we shall show that the slope of the line L_1 is less than the slope of the line L_2. But the slope of L_1 is given by m_1 where

$$m_1 = \frac{\partial y}{\partial x}\bigg|_{(x_0,y_0)} = \frac{(x_0-1)y_0}{(1-y_0)x_0}$$

and the slope of L_2 is given by m_2 where,

$$m_2 = \frac{1-x_0}{y_0-1}.$$

We want to show that

$$m_1 < m_2.$$

But $$m_1 - m_2 = \frac{(x_0-1)y_0}{(1-y_0)x_0} + \frac{x_0-1}{y_0-1}$$

$$= \frac{x_0-1}{(1-y_0)x_0} [y_0-x_0]$$

Now $y_0 > 1$ and if $\beta^* > \pi/4$ then $y_0 > x_0$ which implies that $m_1<m_2$.

Also we must show that the tangent line of the curve G_1 at the point $(x_1,1)$ is vertical but the slope of the tangent line at $(x_1,1)$ is

$$m = \frac{\partial y}{\partial x}\Big|_{(x_1,1)} = \frac{(x-1)y}{(1-y)x}\Big|_{(x_1,1)} = \infty$$

This ends the proof.

Lemma (3.2.3)

Under the condition of Lemma (3.2.2), the Modified L.R.T. has a V-decreasing acceptance region.

Proof:

We want to show that the acceptance region is a decreasing set with respect to the cone V. It is enough to show that, the dual cone V^0 at any point on the boundary of the acceptance region is a subset of the acceptance region.

The boundary of the acceptance region consists of the lines L_2, L_3 and the curve G_1. See Figure 3-2.

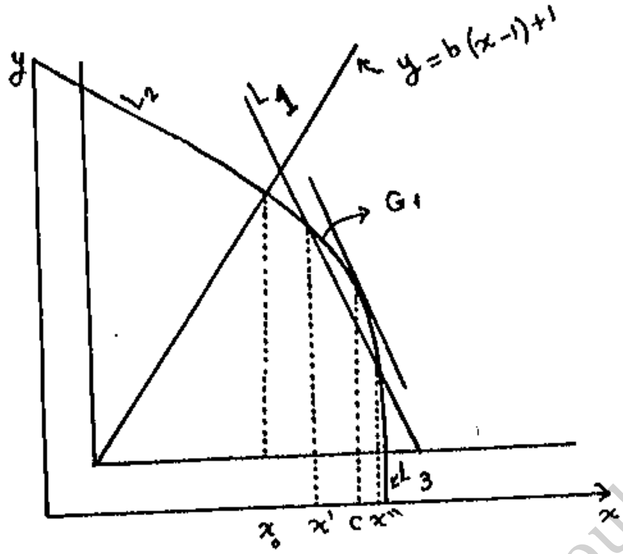

Figure 3-2

It is obvious that the dual cone at any point on the boundary of the lines L_2 and L_3 is a subset of the acceptance region. It is left to show that the dual cone V^0 at any point (x_1, y_1) on the curve G_1 is a subset of the acceptance region. This reduces to show that:

1. The curve G_1 is a decreasing function with respect to x, which was proved by Lemma (3.2.2).
2. Let the line L_4 intersect the curve G_1 in two points (x', y') and (x'', y'') with $x' < x''$. We must show that the slope of L_4 is less than the slope L_2. By the Mean Value Theorem there exists a point $C \in (x', x'')$ such that

$$\left.\frac{\partial y}{\partial x}\right|_{x=C} = \frac{y(x') - y(x'')}{x' - x''}$$

where y(x) is the height of the curve G_1 at x. But we have shown in Lemma (3.2.2) that the curve G_1 is concave down which implies that $\frac{\partial y}{\partial x}$ is a decreasing function of x.

Therefore,

$$\frac{\partial y}{\partial x}\bigg|_{x=x_0} \leq \frac{\partial y}{\partial x}\bigg|_{x=C}$$

which implies that the slope m_4 of the line L_4 is less than the slope m_2 of the line L_2 where

$$m_4 = \frac{\partial y}{\partial x}\bigg|_{x=C}$$

$$m_2 = \frac{\partial y}{\partial x}\bigg|_{x=x_0}$$

This ends the proof.

Theorem (3.2.4)

Under the condition of Lemma (3.2.2), the Modified L.R.T. is an admissible test.

Proof:

Since the Modified L.R.T. has a convex and V-decreasing acceptance region for the case $\pi/4 \leq \beta^* \leq \pi/2$, then by Eaton's (1970) Theorem, we get that the Modified L.R.T. is an admissible test.

Corollary (3.2.5)

Consider the statistical model (1.1.1) and the testing problem (1.1.3). The M.D.L.R.T. has a convex acceptance region.

proof:

It is enough to show that

(1). The two curves G_1 and G_2 are concave dawn functions where G_1, G_2 are shown in the Figure 2-4 .

In lemma (3.2.2), we have shown that the curve G_1 is concave down function .

Now ,we want to show that the curve G_2 is a concave down function in x.

The curve G_2 has the following equation

$$A\ B\ \exp(-xt-ys) = k_1$$

where $t = 1 - (1/A)$,$s = 1 - (1/B)$

$$A = \left[\frac{x+by-b-b^2}{b^2+1}\right] \quad \text{and} \quad B = \left[\frac{bx + b^2y-b+1}{b^2+1}\right]$$

We can easily get that :

$$\partial y/\partial x=-1/b\left[\frac{B(A-1)-D/(b^2+1)}{A(A-1)-D/(b^2+1)}\right] = m_2 \qquad (3.2.1)$$

where $D = B+bA-xB/A-bAy/B$

A mathematical proof for $\frac{\partial^2 y}{\partial x^2}$ to be negative is not available, however we show, through some numerical computations that $\partial y/\partial x$ is negative, and $\partial y/\partial x$ is decreasing. Table (1) in the Appendix shows these computations for the cases $\beta^* = 30^o$, 60^o and 78^o. Now consider the point (x_o,y_o) which is the intersection point between the curve G_1 and the line $y=b(x-1)+1$, then the slope of the tangent line for the curve G_1 at this point is equal to

$$m_1=[(x_o-1)y_o]/[(1-y_o)x_o]$$

Notice that the point (x_o,y_o) is also the point of intersection between the line $y=b(x-1)+1$ and the curve G_2, then the slope of the tangent line for the curve G_2 at this point will be equal to m_2 which is given by (3.2.1)

Now ,we want to show that the $m_1 \le m_2$. Numerically we get the following results :

at $\beta^* = 60$ we have $m_1 = -.8058$, $m_2 = -.4650$

at $\beta^* = 75$ we have $m_1 = -.5548$, $m_2 = -.1490$

at $\beta^* = 30$ we have $m_1 = -1.2409$, $m_2 = -2.1490$

From these numerical results we may conjecture that the M.D.L.R.T. has a convex acceptance region for case that $\beta^* \ge \pi/4$

3.3 Monotonicity and symmetry properties

In this section, we study some properties of the power function of the L.R.T. for the value $\beta^* = \pi/2$, which is, in this case, equal to the power functions of the Modified L.R.T. and M.D.L.R.T. In addition, we prove, for this case, that the power functions of these tests are symmetric about $\pi/4$. Also, they have maximum at $\beta = \pi/2$ and two equal minima at $\beta = 0$ and $\beta = \pi/2$. Furthermore, for some other cases of β^* and through some numerical computations given in section 3.4, it is observed that for fixed Δ, the power function of some test is monotone in β for $\beta < \bar{\beta}$, and it is symmetric about $\bar{\beta}$, where $\bar{\beta}$ is determined by the angle β^* of the cone.

Theorem (3.3.1)

For the case that $\beta^* = \pi/2$, and for fixed Δ, the power function $\mathcal{B}_1(\Delta,\beta)$ is symmetric in β about $\beta = \pi/4$. i.e.,

$$\mathcal{B}_1(\Delta,\pi/2-\beta) = \mathcal{B}_1(\Delta,\beta) \qquad \text{for } \beta \le \pi/4$$

Proof:

The power function $\mathcal{B}_1(\Delta,\beta)$ is given by (2.2.4). First,

$$\mathcal{B}_1(\Delta,\pi/2-\beta) = \frac{1}{\theta_2}\int_1^{C_4}\exp\left[-\frac{y}{\theta_2}-\frac{w_1(y)}{\theta_1}\right]dy + \exp\left[-\frac{C_4}{\theta_2}\right]\left(1-e^{-1/\theta_2}\right)$$

$$+ \exp\left[-\frac{C_4}{\theta_2}\right]$$

where $\qquad \theta_2 = (\theta_1-1)\tan(\frac{\pi}{2}-\beta)+1$

and $w_1(y)$ as defined earlier.

Since the curve G_1 given by

$$\theta_1\exp(-\theta_1) = \Delta/\theta_2\ \exp(\theta_2-2)$$

is symmetric about $\theta_1 = \theta_2$. (see Figure 3-3)

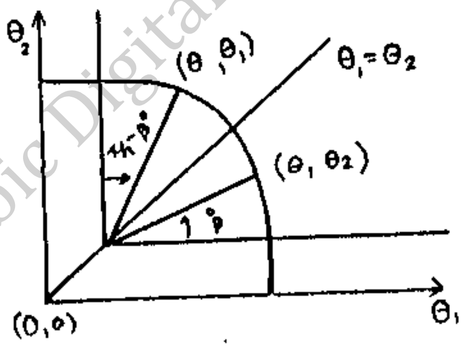

Figure 3-3

Illustration of (θ_1,θ_2) related with $\Delta = \theta_1\ \theta_2\ \exp(-\theta_1-\theta_2+2)$

Then,

$$\tan\beta = \frac{\theta_2-1}{\theta_1-1} \qquad \text{and} \qquad \tan(\frac{\pi}{2}-\beta) = \frac{\theta_2-1}{\theta_1-1}$$

Thus,

$$\mathcal{B}_1(\Delta, \pi/2-\beta) = \mathcal{B}_1(\Delta,\beta) \qquad \text{for } \beta \le \pi/4$$

This ends the proof.

Theorem (3.3.2)

For the case $\beta^* = \pi/2$, the power function has a maximum point at $\beta = \pi/4$.

Proof:

The power function is given by

$$B(\theta_1,\theta_2) = 1-\int_A\int \frac{1}{\theta_1} e^{-x/\theta_1} \frac{1}{\theta_2} e^{-y/\theta_2} \, dx\, dy$$

This can be written in term of (Δ,β) as

$$\eta_\Delta(\beta)=\mathcal{B}_1(\Delta,\beta)=1-\iint_A \frac{1}{\theta_1}e^{-x/\theta_1} \frac{1}{(\theta_1-1)\tan\beta+1}e^{-y/[(\theta_1-1)\tan\beta+1]} dxdy$$

Where (Δ,β) are related to (θ_1,θ_2) as in (2.2.2).

and the region A is illustrated in Figure 3-4.

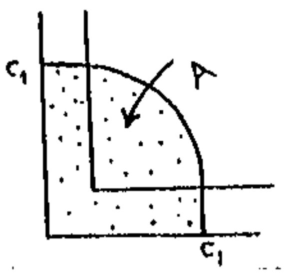

Figure 3-4

Sketch of the acceptance region of L.R.T. ϕ'_{V_1}

Now,

$$\frac{\partial\eta_\Delta(\beta)}{\partial\beta} = -\left[\iint_A \frac{1}{\theta_1}e^{-x/\theta_1}\, e^{-y/[(\theta_1-1)\tan\beta+1]} H^*(x,y,\theta_1,\beta)\, dxdy\right] \qquad (3.3.1)$$

where

$$H^*(x,y,\theta_1,\beta) = \left(\frac{\partial}{\partial\beta} g(\theta_1,\beta)\right)(1-yg(\theta_1,\beta)+\left(\frac{\partial\theta_1}{\partial\beta}\right)\left(\frac{x}{\theta_1^2} - \frac{1}{\theta_1}\right) g(\theta_1,\beta),$$

$$g(\theta_1,\beta) = 1 \;/\; [(\theta_1-1)\tan\beta + 1]$$

$$\frac{\partial}{\partial\beta} g(\theta_1,\beta) = -g^2(\theta_1,\beta) \left[(\theta_1-1) \sec^2\beta + \frac{\partial\theta_1}{\partial\beta} \tan\beta\right]$$

and

$$\frac{\partial\theta_1}{\partial\beta} = \frac{(1-\theta_1)\sec^2\beta\; (\theta_1-\Delta e^{(\theta_1-1)(1+\tan\beta)})}{1+(2\theta_1-1)\tan\beta - \Delta(1+\tan\beta)\; e^{(\theta_1-1)(1+\tan\beta)}}$$

Now, at $\beta = \pi/4$, we get $\frac{\partial\theta_1}{\partial\beta} = (1-\theta_1)$

Therefore

$$\frac{\partial\eta_\Delta(\frac{\pi}{4})}{\partial\beta} = -\iint_A \frac{1}{\theta_1} e^{-x/\theta_1-y/[(\theta_1-1)\tan\beta+1]} \left\{-\frac{[2(\theta_1-1)+(1-\theta_1)]}{\theta_1^2}\,\frac{\theta_1-y}{\theta_1} + \frac{1-\theta_1}{\theta_1}\,\frac{x-\theta_1}{\theta_1}\,\frac{1}{\theta_1}\right\} dx\,dy$$

$$= -\iint_A \frac{1}{\theta_1} e^{-x/\theta_1-y/[(\theta_1-1)\tan\beta+1]}\, \frac{1}{\theta_1^3}[-(\theta_1-1)(\theta_1-y)+(1-\theta_1)(x-\theta_1)]dxdy$$

$$= \frac{(1-\theta_1)}{\theta_1^4} \iint_A e^{-x/\theta_1-y/[(\theta_1-1)\tan\beta+1]}\; [x-y]\; dx\; dy$$

since there is symmetry about $x = y$, $\frac{\partial\eta_\Delta(\pi/4)}{\partial\beta} = 0$. We can see from the table (3) of the power function that at $\beta=\pi/4$, in fact, it has maximum. This ends the proof.

Theorem (3.3.3)

For the case $\beta^* = \pi/2$, the power function has two local minima at $\beta = 0$ and $\beta = \pi/2$.

Proof:

Using Formula (3.3.1), we have for $\beta = 0$

$$\frac{\partial\theta_1}{\partial\beta} = \frac{(1-\theta_1)(\theta_1-\Delta e^{\theta_1-1})}{1-\Delta e^{\theta_1-1}}$$

$$\frac{\partial}{\partial\beta} g(\theta_1,\beta) = (1-\theta_1) \text{ and}$$

$$\Delta = \theta_1 e^{-\theta_1+1}$$

Thus

$$\frac{\partial}{\partial\beta}\eta_\Delta(0) = -\iint_A \frac{1}{\theta_1} e^{-x/\theta_1-y} [(1-\theta_1)(1-y)]\, dx\, dy$$

$$= \frac{(\theta_1-1)}{\theta_1}\int\int_A e^{-x/\theta_1-y}(1-y)\, dx\, dy$$

$$= \frac{(\theta_1-1)}{\theta_1}\left[\int_0^{C_1}\int_0^1 e^{-x/\theta_1-y}(1-y)dxdy + \int_1^{C_1}\int_0^{L(x)} e^{-x/\theta_1-y}(1-y)dxdy\right]$$

where C_1 is the solution of $C_1e^{-C_1} = k_1'' e^{-1}$ with $C_1 \ge 1$ and $y = L(x)$ is the curve given by the solution for y of the following equation

$$y\, e^{-y} = \frac{k_1''}{x} e^{x-2}$$

Now

$$\frac{\partial}{\partial\beta}\eta_\Delta(0) = (\theta_1-1)[(1-e^{-1/\theta_1})(C_1e^{-C_1}) + \int_1^{C_1} e^{-x/\theta_1} [L(x)e^{-L(x)}]\, dx$$

$$= (\theta_1-1)[(1-e^{-1/\theta_1})(C_1e^{-C_1}) + \int_1^{C_1} \frac{k_1'' e^{-2}}{x} e^{-x(1-1/\theta_1)}\, dx$$

$$= (\theta_1-1)[(1-e^{-1/\theta_1})(C_1e^{-C_1}) + k_1'' e^{-2}\int_1^{C_1} \frac{e^{-x(1-1/\theta_1)}}{x}\, dx$$

Since $\theta_1 > 0$, $0 < k_1'' < 1$ and $\int_1^{C_1} \frac{e^{-x(1-1/\theta_1)}}{x} dx > 0$

then $\frac{\partial}{\partial\beta} \eta_\Delta(0) \geq 0$,

Since the range of β start from zero, and the derivative is positive at β=0 then it has local minimum of β=0 gives at β = 0, there is a local minimum of the power function.

Since the power function is symmetric about $\beta=\pi/4$, then it has another minimum point at $\beta=\pi/2$. This ends the proof.

3.4 Numerical results

In this section, are given some numerical computations to illustrate the properties of the power functions for the different tests considered earlier. These numerical computations are taken for certain cones of angles $\beta^*=2\pi$, $\pi/2$, $\pi/3$, $\pi/4$ and $\pi/6$. We showed that these tests are coincide for $\beta^*=\pi/4$ and $\beta^*=\pi/2$, therfore when we speak about these two cases we deal with one test which is equivalent to the other tests. The results for these computations are given in Tables (2),...., (9) of the Appendix. From these tables we observe the following:

Observation (1): For the case that $\beta^*=\pi/2$ and for fixed Δ, the power function $\mathcal{B}_1(\Delta,\beta)$ is an increasing function in β for $0\leq\beta\leq\pi/4$ and a decreasing function for $\pi/4\leq\beta<\pi/2$.

To prove this, we take $\frac{\partial\eta_\Delta(\beta)}{\partial\beta}$ which equal to the form given by (3.3.1). We leave the proof of this observation for future research.

<u>Observation (2):</u> For the case $\beta^*=\pi/2$ and fixed Δ, the power function $\mathcal{B}_1(\Delta,\beta)$ is symmetric in β about $\pi/4$. In other words

$$\mathcal{B}_1(\Delta,\pi/2-\beta) = \mathcal{B}_1(\Delta,\beta) \qquad \text{for } \beta \le \pi/4$$

Which was proved in Theorem (3.3.1)

<u>Observation (3):</u> For the case $\beta^*=\pi/2$ and for fixed Δ, the power function $\mathcal{B}_1(\Delta,\beta)$ has a maximum point at $\pi/4$, and two equal minima at $\beta=0$, and $\beta=\pi/2$, which was partially proved in Theorem (3.3.2) and Theorem (3.3.3).

<u>Observation (4):</u> For the case that $0\le\beta^*\le\pi/2$ and for fixed β, the power function is a decreasing in Δ.

<u>Observation (5):</u> For the case $\beta^*=\pi/4$, $\pi/3$ and $\pi/6$, for different Δ, the power function is an increasing function in β for $0\le\beta\le\bar{\beta}$ and a decreasing function for $\bar{\beta}\le\beta\le\beta^*$, where $\bar{\beta}$ is determined by the cone V with angle β^* and varies with Δ .

The following tables give the point $\bar{\beta}$, the increasing interval and the decreasing interval for all cases.

(1) For $\beta^{*}= \pi/4$.

Δ	Symmetric point	Increasing interval	Decreasing interval
0.1	39^{o}	$[0^{o},39^{o}]$	$[39^{o},45^{o}]$
0.3	36^{o}	$[0^{o},36^{o}]$	$[36^{o},45^{o}]$
0.5	33^{o}	$[0^{o},33^{o}]$	$[33^{o},45^{o}]$
0.7	30.5^{o}	$[0^{o},30.5^{o}]$	$[30.5^{o},45^{o}]$
0.9	29^{o}	$[0^{o},29^{o}]$	$[29^{o},45^{o}]$

(2) For $\beta^{*}= \pi/6$ (Modefied L.R.T.)

Δ	Symmetric point	Increasing interval	Decreasing interval
.1	——	$[0,30^{o}]$	——
.3	22^{o}	$[0,22^{o}]$	$[22^{o},30^{o}]$
.5	19^{o}	$[0,19^{o}]$	$[19^{o},30^{o}]$
.7	16^{o}	$[0,16^{o}]$	$[16^{o},30^{o}]$
.9	15.5^{o}	$[0,15.5^{o}]$	$[15.5^{o},30]$

(3) For $\beta^{*}=\pi/6$ (M.D.L.R.T.)

Δ	Symmetric point	Increasing interval	Decreasing interval
.1	——	$[0,30^{o}]$	——
.3	——	$[0,30^{o}]$	——
.5	26^{o}	$[0,26^{o}]$	$[26^{o},30^{o}]$
.7	23^{o}	$[0,23^{o}]$	$[23^{o},30^{o}]$
.9	21.5^{o}	$[0,21.5^{o}]$	$[21.5^{o},30^{o}]$

(4) For $\beta^{*} = \pi/3$ (Modefied L.R.T.)

Δ	Symmetric point	Increasing interval	Decreasing interval
.1	42^{o}	$[0, 42^{o}]$	$[42^{o}, 60^{o}]$
.3	40^{o}	$[0, 40^{o}]$	$[40^{o}, 60^{o}]$
.5	38.5^{o}	$[0, 38.5^{o}]$	$[38.5^{o}, 60^{o}]$
.7	36^{o}	$[0, 36^{o}]$	$[36^{o}, 60^{o}]$
.9	35^{o}	$[0, 35^{o}]$	$[35^{o}, 60^{o}]$

(5) For $\beta^{*} = \pi/3$ (M.D.L.R.T.)

Δ	Symmetric point	Increasing interval	Decreasing interval
.1	42^{o}	$[0, 42^{o}]$	$[42^{o}, 60^{o}]$
.3	38^{o}	$[0, 38^{o}]$	$[38^{o}, 60^{o}]$
.5	36^{o}	$[0, 36^{o}]$	$[36^{o}, 60^{o}]$
.7	35^{o}	$[0, 35^{o}]$	$[35^{o}, 60^{o}]$
.9	32.5^{o}	$[0, 32.5^{o}]$	$[32.5^{o}, 60^{o}]$

CHAPTER FOUR

NUMERICAL COMPARISON

4.1 Introduction

This chapter is devoted to give some numerical comparison between the different power functions of the L.R.T. the M.D.L.R.T. and the Modified L.R.T.. These tests coincide at $\beta^*=\frac{\pi}{4}$ and $\frac{\pi}{2}$. It is observed that the power of these tests are closed to each other. Also, some of the properties presented in Chapter 3 can be observed.

4.2 Basic Results

This section contains a condition for which , the two tests ϕ_1, the Modified L.R.T. and ϕ_2 the L.R.T. have the same level of significance which is in turn the necessary condition for one of them to dominates the other . But the level of significance for the test are $\mathcal{B}_1(1,\beta)$ and $\mathcal{B}_2(1,\beta)$ where $\mathcal{B}_i(\Delta,\beta)$ is the power function of the test ϕ_i .

The condition is

$$\mathcal{B}_1(1,\beta) = \mathcal{B}_2(1,\beta) \qquad \forall\ \beta$$

which is equivalent to

$$\int_1^{b(A-1)+1} e^{-y-w_1(y)}dy + e^{-A-[b(A-1)+1]} + \frac{b}{b-1}\,e^{-c}\left(1-e^{-A+A/b}\right)$$

$$+ e^{-C_4}(1-e^{-1}) = 1+e^{-1-C_1} + \int_1^{C_1} e^{-y-w_2(y)} - e^{-1-(2-c_1)^+} + \int_{(2-C_1)^+}^{1} e^{-x-w_3(x)}dy$$

$$- \int_{(2-C_1)^+}^{1} e^{-y-(2-\bar{w}_2(y))^+} dy + \int_{(2-C_1)^+}^{1} e^{-y-\bar{w}_2(y)} dy \qquad (4.2.2)$$

where b, A, C, C_4, H^+, $w_1(y)$, $w_2(y)$, and $\bar{w}_2(y)$ are defined earlier in section 2.2.

4.3 Computational Examples

In this section, we get the power for the three tests for five values of β^* which are $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$ and 2π.

Now, for $\beta^* = \frac{\pi}{4}$, $\frac{\pi}{2}$, and $\beta^* = 2\pi$, the tests M.D.L.R.T., L.R.T. ,and Modified L.R.T. are coincide.

For $\beta^* = \frac{\pi}{3}$ and $\frac{\pi}{6}$ we find the power functions of the Modified L.R.T. and for the M.D.L.R.T., we observe that the power function of these tests are closed to each others.

Assume that ϕ_{V_i} be the modified L.R.T. corresponding for $P(V_i)$ problem for i = 1, 2, 3, 4 and 5 where,

$P(V_i)$: $H_o: (\theta_1, \theta_2) = (1,1)$ against $H_o: (\theta_1, \theta_2) = \in V_i - \{(1,1)\}$ and V_i i=1,...,5 are closed convex cone with angles $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, and 2π, respectively.

Note that ϕ_{V_2}, ϕ_{V_4} and ϕ_{V_5} are equivalent to M.D.L.R.T. and L.R.T.. Let K_i's are constants based on the level of significance α which is chosen to be 0.05. Notice that we want to solve the equations, in K_i, which are,

$$E_{(1,1)}\, \phi_{V_i} = .05 \qquad \text{for } i = 1,2,3,4 \text{ and } 5$$

, where

$$\phi v_i = \begin{cases} 1 & \Lambda_i'' < k_i \\ 0 & \Lambda_i'' \geq k_i \end{cases}$$

where Λ'''s (i=1,2,3, and 4) are defined by (2.1.8) based on the cone angle β^*, and $\Lambda_5'' = x\, y\, e^{-x-y+2}$.

We can obtain the constants k_i's (i=1,2,3 and 4), by solving the following equations:

$$\int_1^{C_2^*/2} e^{-y-w_1(y)}\, dy + e^{-C_2^*} + e^{-C_2}(1-e^{-1}) = 0.05 \quad \text{for } i=2$$

$$\int_1^{b(A-1)+1} e^{-y-w_1(y)} dy + e^{-A-[b(A-1)+1]} + \frac{b}{b-1} e^{-C_i^*}\left(1-e^{-A+A/b}\right)$$

$$+ e^{-C_i}(1-e^{-1}) = 0.05 \quad \text{for } i = 1,\ 3 \text{ and } 4$$

and

$$1 + e^{-1-C_5} + \int_1^{C_5} e^{-y-w_2(y)} dy + e^{-1-(2-C_5)^+} + \int_{(2-C_5)^+}^{1} e^{-x-w_3(x)} dx$$

$$- \int_{(2-C_5)^+}^{1} e^{-y-(2-\bar{w}_2(y))^+} dy + \int_{(2-C_5)^+}^{1} e^{-y-\bar{w}_2(y)} dy = 0.05 \quad \text{for } i=5 \qquad (4.3.3)$$

where C_i's be the solution of $C_i\, e^{-C_i} = k_i\, e^{-1}$ such that

$$C_i > 1 \quad (i = 1,\ 2,\ 3,\ 4 \text{ and } 5)$$

and C_i^*'s are related by

$$A\, e^{-A} = k_i \frac{e^{[b(A-1)+1]-2}}{b(A-1)+1} \quad \text{such that}$$

$$A = b(b + C_i^* - 1) / (b^2+1)$$

Also, we can easily obtain the constants k_{1i} and k_{2i} (i=1,3) of the M.D.L.R.T. by solving the following equations:

$$\bar{B}_i(\Delta,\beta) = 0.05, \qquad i=1, \text{ and } 3$$

where $\bar{B}_i(\Delta,\beta)$ is defined by (2.2.5).

We get using the relations $K_{2i}=C_{2i}e^{1-C_{2i}}$ for i=1,3, that $K_{21}=0.3030$, $K_{23}=0.2655$, $C_{21}=3.4282$, and $C_{23}=3.6098$.

We get the values of k_i's (i=1,2,3 and 4) are 0.3259, 0.2913, 0.2649, and 0.2361 respectively. Therefore, the values of C_i's (i=1,2,3 and 4) are 3.3216, 3.4806, 3.6131 and 3.7707 respectively. Also, the values of C_i^*'s (i=1,2,3 and 4) are 6.0013, 5.1091, 4.5001, and 3.7707 respectively.

Now, since the value of C_4 = 3.7707 then the value of C_5 must be greater than C_4. Further, since $C_5 > 2$ then the formula (4.3.3) can be written as:

$$e^{-1-C_5'} + \int_1^{C_5} e^{-y-w_2(y)}dy + \int_0^1 e^{-x-w_3(x)}dx + \int_0^1 e^{-y-\bar{w}_2(y)}dy = 0.05. \tag{4.3.4}$$

This equation (4.3.4) obtained from $B_2(1,1)$ under the case that $C_5 > 2$. Thus, we have k_5 = 0.2132 and C_5 = 3.9087.

For further comparison, we deal with the M.D.L.R.T. under $\beta^* = \frac{\pi}{3}$, and $\frac{\pi}{6}$. We observed that this test is nearly closed to the Modified L.R.T..

Now, for the L.R.T., as we said before, there is some difficulty to deal with curve G_2 which is $Q(x,y) = K_1'$. This curve depend on solution of the cubic equation which is not easy to deal with.

For all the above reasons, we defined two other tests which are called Modified L.R.T. and M.D.L.R.T., which closed to the L.R.T. under certain values. From Tables 2, 3 and 4 of the Appendix, we observe that for fixed Δ the power function for the restricted alternative with $\beta^* = \frac{\pi}{4}$ is greater than other power function for different cone angles $\frac{\pi}{2}$ and 2π. For $\beta^* = \frac{\pi}{3}$ and $\frac{\pi}{6}$, the power function of the M.D.L.R.T. is nearly close to the power of the Modified L.R.T.. But for these tests we observed that there is no existence for that more restrictions on the alternative space give more powerful test, so at $\beta^* = \pi/4$ we get more powerful test.

In subinterval of β and for the M.D.L.R.T., we get that the powere function at $\beta^* = \pi/6$ is better than the power function at $\beta^* = \pi/3$. Also for the Modified L.R.T., in subinterval of β, the power function at $\beta^* = \pi/6$ is better than the power function at $\beta^* = \pi/3$.

The L.R.T. has the property that the more restrictions on the alternative space will give more powerful test. Also, we conclude that the L.R.T. ϕ_{V_8} is an inadmissible test for the problem $P(V_i)$; i=2 and 4. In general the L.R.T. ϕ_{V_i} is an inadmissible test for the problems $P(V_j)$ where $j < i$, $i,j=2,4$ and 8.

APPENDIX

TABLE 1

The derivative of the curve G_2 for $\beta^* = 30^o$, 60^o and 75^o.
where G_2 is the boundary of the acceptance
region in the cone V_1^+

	$\beta^* = 30^o$		$\beta^* = 60^o$		$\beta^* = 75^o$	
x	y	y′	y	y′	y	y′
0.10	6.041	-2.406	4.768	-0.496	4.325	-0.171
0.20	5.903	-2.398	4.682	-0.495	4.261	-0.170
0.30	5.764	-2.390	4.596	-0.493	4.198	-0.168
0.40	5.627	-2.381	4.511	-0.492	4.135	-0.167
0.50	5.490	-2.372	4.426	-0.490	4.073	-0.165
0.60	5.353	-2.364	4.341	-0.489	4.012	-0.164
0.70	5.217	-2.355	4.257	-0.487	3.951	-0.162
0.80	5.081	-2.346	4.172	-0.486	3.891	-0.161
0.90	4.946	-2.337	4.088	-0.484	3.831	-0.159
1.00	4.811	-2.329	4.005	-0.483	3.772	-0.158
1.10	4.677	-2.320	3.921	-0.481	3.714	-0.156
1.20	4.543	-2.311	3.838	-0.480	3.656	-0.155
1.30	4.410	-2.302	3.755	-0.478	3.598	-0.153
1.40	4.277	-2.293	3.672	-0.477	3.541	-0.152
1.50	4.145	-2.284	3.590	-0.475	3.485	-0.151
1.60	4.014	-2.275	3.508	-0.474	3.429	-0.149
1.70	3.883	-2.266	3.426	-0.472		
1.80	3.752	-2.258	3.344	-0.471		
1.90	3.622	-2.249	3.263	-0.469		
2.00	3.492	-2.240	3.181	-0.468		
2.10	3.363	-2.231	3.101	-0.466		
2.20	3.235	-2.222				
2.30	3.107	-2.213				
2.40	2.979	-2.204				
2.50	2.852	-2.196				
2.60	2.726	-2.187				
2.70	2.600	-2.178				
2.80	2.474	-2.170				
2.90	2.349	-2.161				
3.00	2.225	-2.153				

Notice that the range of the curve G_2 varies with β^*, for this reason there are missing values for the case of $\beta^*=60^o$ and $\beta^*=75^o$

TABLE 2

The power function of the L.R.T. for the case $\beta^* = \pi/4$

β	value of Δ .1	.2	.3	.4	.5	.6	.7	.8	.9
1	.5202	.4495	.3956	.3487	.3054	.2638	.2223	.1794	.1320
2	.5243	.4530	.3985	.3513	.3076	.2656	.2238	.1805	.1327
3	.5282	.4562	.4013	.3537	.3097	.2673	.2252	.1816	.1334
4	.5319	.4594	.4040	.3560	.3117	.2690	.2265	.1826	.1341
5	.5355	.4624	.4066	.3582	.3135	.2706	.2278	.1835	.1347
6	.5390	.4652	.4090	.3603	.3153	.2721	.2290	.1845	.1353
7	.5422	.4679	.4114	.3623	.3170	.2735	.2301	.1853	.1358
8	.5454	.4705	.4136	.3642	.3186	.2748	.2312	.1861	.1364
9	.5484	.4730	.4157	.3660	.3201	.2761	.2322	.1869	.1369
10	.5512	.4754	.4177	.3677	.3216	.2773	.2332	.1876	.1373
11	.5539	.4776	.4195	.3693	.3229	.2784	.2341	.1883	.1378
12	.5564	.4797	.4213	.3708	.3242	.2794	.2350	.1890	.1382
13	.5588	.4817	.4230	.3722	.3254	.2804	.2357	.1896	.1386
14	.5611	.4836	.4246	.3735	.3265	.2813	.2365	.1901	.1390
15	.5633	.4854	.4261	.3748	.3275	.2822	.2372	.1906	.1393
16	.5653	.4871	.4275	.3760	.3285	.2830	.2378	.1911	.1396
17	.5672	.4887	.4288	.3771	.3294	.2837	.2384	.1915	.1399
18	.5690	.4901	.4301	.3781	.3303	.2844	.2389	.1919	.1401
19	.5706	.4915	.4312	.3791	.3311	.2850	.2394	.1923	.1404
20	.5722	.4928	.4323	.3799	.3318	.2856	.2398	.1926	.1406
21	.5737	.4940	.4333	.3808	.3324	.2861	.2402	.1929	.1408
22	.5750	.4951	.4342	.3815	.3330	.2866	.2406	.1932	.1409
23	.5763	.4962	.4350	.3822	.3336	.2870	.2409	.1934	.1411
24	.5774	.4971	.4358	.3828	.3341	.2874	.2412	.1936	.1412
25	.5785	.4980	.4365	.3834	.3345	.2877	.2414	.1937	.1413
26	.5795	.4988	.4372	.3839	.3349	.2880	.2416	.1938	.1413
27	.5804	.4995	.4377	.3843	.3352	.2882	.2417	.1939	.1414
28	.5812	.5001	.4382	.3847	.3355	.2884	.2419	.1940	.1414
29	.5820	.5007	.4387	.3850	.3357	.2885	.2419	.1940	.1414
30	.5826	.5012	.4390	.3853	.3359	.2886	.2420	.1940	.1414
31	.5832	.5016	.4393	.3855	.3360	.2887	.2420	.1940	.1414
32	.5837	.5020	.4396	.3856	.3361	.2887	.2419	.1939	.1413
33	.5842	.5023	.4398	.3857	.3361	.2887	.2419	.1939	.1412
34	.5845	.5026	.4399	.3858	.3361	.2886	.2418	.1937	.1411
35	.5848	.5027	.4400	.3858	.3360	.2885	.2416	.1936	.1410
36	.5851	.5028	.4400	.3857	.3359	.2883	.2415	.1934	.1409
37	.5852	.5029	.4400	.3856	.3357	.2882	.2413	.1932	.1407
38	.5853	.5029	.4399	.3855	.3355	.2879	.2410	.1930	.1406
39	.5854	.5028	.4397	.3852	.3353	.2877	.2407	.1927	.1404
40	.5853	.5027	.4395	.3850	.3350	.2873	.2404	.1925	.1402
41	.5852	.5025	.4393	.3847	.3347	.2870	.2401	.1922	.1399
42	.5851	.5022	.4390	.3843	.3343	.2866	.2397	.1918	.1397
43	.5848	.5019	.4386	.3839	.3338	.2862	.2393	.1915	.1394
44	.5845	.5015	.4382	.3835	.3334	.2857	.2389	.1911	.1391

Note: This table gives the powers of the M.D.L.R.T. and modified L.R.T. since they are equivelant to the L.R.T. for $\beta^* = \pi/4$

TABLE 3

The power function of the L.R.T. for

the case $\beta^{*}=\pi/2$

β	value of Δ .1	.2	.3	.4	.5	.6	.7	.8	.9
1	.4949	.4233	.3694	.3232	.2810	.2410	.2018	.1620	.1193
3	.5043	.4313	.3763	.3292	.2862	.2454	.2054	.1648	.1211
5	.5131	.4387	.3827	.3348	.2910	.2494	.2087	.1673	.1228
7	.5213	.4457	.3887	.3399	.2954	.2532	.2118	.1697	.1244
9	.5289	.4521	.3943	.3447	.2996	.2567	.2147	.1719	.1259
11	.5358	.4580	.3994	.3492	.3034	.2600	.2173	.1740	.1272
13	.5421	.4634	.4041	.3533	.3069	.2629	.2198	.1759	.1285
15	.5478	.4683	.4084	.3570	.3101	.2657	.2221	.1777	.1297
17	.5529	.4728	.4123	.3604	.3131	.2682	.2241	.1793	.1308
19	.5574	.4768	.4158	.3635	.3158	.2705	.2260	.1808	.1319
21	.5615	.4804	.4190	.3663	.3182	.2726	.2278	.1822	.1328
23	.5651	.4836	.4218	.3688	.3204	.2744	.2293	.1834	.1336
25	.5683	.4865	.4244	.3711	.3223	.2761	.2307	.1845	.1344
27	.5711	.4890	.4267	.3730	.3241	.2776	.2320	.1855	.1351
29	.5735	.4912	.4286	.3748	.3255	.2789	.2330	.1864	.1357
31	.5755	.4931	.4303	.3763	.3269	.2800	.2340	.1871	.1362
33	.5774	.4947	.4318	.3776	.3281	.2810	.2348	.1878	.1367
35	.5789	.4961	.4330	.3787	.3290	.2818	.2355	.1883	.1371
37	.5801	.4972	.4340	.3796	.3298	.2825	.2361	.1888	.1374
39	.5810	.4980	.4348	.3802	.3304	.2830	.2365	.1891	.1376
41	.5816	.4986	.4353	.3807	.3308	.2834	.2368	.1893	.1378
43	.5820	.4990	.4356	.3810	.3311	.2836	.2370	.1895	.1379
45	.5821	.4991	.4357	.3811	.3312	.2837	.2370	.1895	.1379
47	.5820	.4990	.4356	.3810	.3311	.2836	.2370	.1895	.1379
49	.5816	.4986	.4353	.3807	.3308	.2834	.2368	.1893	.1378
51	.5810	.4980	.4348	.3803	.3304	.2830	.2365	.1891	.1376
53	.5801	.4972	.4340	.3796	.3298	.2825	.2361	.1888	.1374
55	.5789	.4961	.4330	.3787	.3291	.2819	.2355	.1883	.1371
57	.5774	.4948	.4318	.3776	.3281	.2811	.2349	.1878	.1367
59	.5757	.4931	.4304	.3764	.3270	.2801	.2340	.1871	.1362
61	.5736	.4912	.4287	.3748	.3257	.2789	.2331	.1864	.1357
63	.5712	.4890	.4267	.3731	.3241	.2776	.2320	.1855	.1351
65	.5684	.4865	.4245	.3711	.3224	.2762	.2308	.1845	.1344
67	.5652	.4837	.4219	.3689	.3205	.2745	.2294	.1834	.1337
69	.5616	.4804	.4191	.3664	.3183	.2726	.2278	.1822	.1328
71	.5575	.4768	.4159	.3636	.3159	.2706	.2261	.1809	.1319
73	.5530	.4728	.4124	.3605	.3132	.2683	.2242	.1794	.1309
75	.5479	.4684	.4085	.3571	.3102	.2658	.2221	.1778	.1298
77	.5422	.4635	.4042	.3534	.3070	.2630	.2199	.1760	.1286
79	.5359	.4581	.3995	.3493	.3035	.2601	.2174	.1741	.1273
81	.5290	.4522	.3944	.3449	.2997	.2568	.2148	.1720	.1259
83	.5214	.4458	.3889	.3401	.2956	.2533	.2119	.1698	.1244
85	.5132	.4389	.3829	.3349	.2911	.2495	.2088	.1674	.1228
87	.5044	.4314	.3765	.3293	.2863	.2455	.2055	.1648	.1212
89	.4950	.4235	.3696	.3234	.2812	.2411	.2019	.1621	.1194

Note: This table gives the powers of the M.D.L.R.T. and modified L.R.T. since they are equivelant to the L.R.T. for $\beta^{*}=\pi/2$

TABLE 4

The power function of the L.R.T. for the case $\beta^{*}=2\pi$

β	value of Δ: .1	.2	.3	.4	.5	.6	.7	.8	.9
1	.4923	.4207	.3668	.3206	.2785	.2386	.1995	.1599	.1175
2	.4965	.4242	.3699	.3233	.2808	.2406	.2011	.1612	.1184
3	.5006	.4277	.3729	.3259	.2831	.2425	.2027	.1624	.1192
4	.5047	.4311	.3758	.3285	.2853	.2444	.2043	.1636	.1200
5	.5086	.4344	.3787	.3309	.2874	.2462	.2057	.1648	.1208
6	.5125	.4377	.3814	.3333	.2895	.2479	.2072	.1659	.1215
7	.5162	.4408	.3841	.3355	.2915	.2496	.2086	.1670	.1222
8	.5198	.4438	.3867	.3378	.2934	.2512	.2099	.1680	.1229
9	.5233	.4467	.3892	.3400	.2952	.2528	.2112	.1690	.1236
10	.5266	.4495	.3916	.3421	.2970	.2543	.2124	.1700	.1243
11	.5298	.4522	.3939	.3440	.2987	.2557	.2136	.1709	.1249
12	.5328	.4547	.3961	.3460	.3003	.2571	.2148	.1718	.1255
13	.5357	.4572	.3982	.3478	.3019	.2584	.2159	.1727	.1261
14	.5385	.4596	.4003	.3495	.3034	.2597	.2169	.1735	.1267
15	.5411	.4618	.4022	.3512	.3048	.2609	.2179	.1743	.1272
16	.5437	.4640	.4041	.3528	.3062	.2621	.2189	.1750	.1277
17	.5460	.4660	.4058	.3544	.3076	.2632	.2198	.1758	.1282
18	.5483	.4680	.4075	.3558	.3088	.2643	.2207	.1765	.1287
19	.5504	.4698	.4091	.3572	.3100	.2653	.2216	.1771	.1292
20	.5524	.4716	.4107	.3586	.3112	.2663	.2224	.1778	.1296
21	.5543	.4732	.4121	.3598	.3123	.2672	.2231	.1784	.1300
22	.5561	.4748	.4135	.3611	.3133	.2681	.2239	.1789	.1304
23	.5578	.4763	.4148	.3622	.3143	.2689	.2245	.1795	.1308
24	.5594	.4777	.4161	.3633	.3152	.2697	.2252	.1800	.1312
25	.5609	.4790	.4172	.3643	.3161	.2705	.2258	.1805	.1315
26	.5623	.4803	.4183	.3653	.3169	.2712	.2264	.1810	.1318
27	.5636	.4815	.4194	.3662	.3177	.2718	.2270	.1814	.1322
28	.5648	.4825	.4203	.3670	.3184	.2725	.2275	.1818	.1324
29	.5660	.4836	.4212	.3678	.3191	.2731	.2280	.1822	.1327
30	.5670	.4845	.4221	.3685	.3198	.2736	.2284	.1826	.1330
31	.5680	.4854	.4229	.3692	.3204	.2741	.2289	.1829	.1332
32	.5689	.4862	.4236	.3699	.3209	.2746	.2292	.1832	.1334
33	.5697	.4870	.4242	.3704	.3214	.2750	.2296	.1835	.1336
34	.5705	.4876	.4248	.3710	.3219	.2754	.2299	.1838	.1338
35	.5712	.4883	.4254	.3715	.3223	.2758	.2302	.1840	.1340
36	.5718	.4888	.4259	.3719	.3227	.2761	.2305	.1842	.1341
37	.5723	.4893	.4263	.3723	.3230	.2764	.2307	.1844	.1343
38	.5728	.4897	.4267	.3726	.3233	.2766	.2310	.1846	.1344
39	.5732	.4901	.4271	.3729	.3236	.2769	.2311	.1847	.1345
40	.5736	.4904	.4273	.3732	.3238	.2771	.2313	.1848	.1346
41	.5739	.4907	.4276	.3734	.3240	.2772	.2314	.1849	.1346
42	.5741	.4909	.4278	.3735	.3241	.2773	.2315	.1850	.1347
43	.5742	.4910	.4279	.3737	.3242	.2774	.2316	.1851	.1347
44	.5743	.4911	.4280	.3737	.3243	.2775	.2316	.1851	.1348
45	.5744	.4911	.4280	.3737	.3243	.2775	.2317	.1851	.1348

TABLE 4, cont.

β	value of Δ .1	.2	.3	.4	.5	.6	.7	.8	.9
46	.5743	.4911	.4280	.3737	.3243	.2775	.2316	.1851	.1348
47	.5742	.4910	.4279	.3737	.3242	.2774	.2316	.1851	.1347
48	.5741	.4909	.4278	.3735	.3241	.2773	.2315	.1850	.1347
49	.5739	.4907	.4276	.3734	.3240	.2772	.2314	.1850	.1347
50	.5736	.4904	.4274	.3732	.3238	.2771	.2313	.1849	.1346
51	.5732	.4901	.4271	.3729	.3236	.2769	.2312	.1847	.1345
52	.5728	.4898	.4267	.3727	.3234	.2767	.2310	.1846	.1344
53	.5724	.4893	.4264	.3723	.3231	.2764	.2308	.1844	.1343
54	.5718	.4888	.4259	.3719	.3227	.2761	.2305	.1842	.1341
55	.5712	.4883	.4254	.3715	.3224	.2758	.2303	.1840	.1340
56	.5705	.4877	.4249	.3710	.3219	.2754	.2300	.1838	.1338
57	.5698	.4870	.4243	.3705	.3215	.2751	.2296	.1835	.1336
58	.5689	.4862	.4236	.3699	.3210	.2746	.2293	.1832	.1334
59	.5680	.4854	.4229	.3693	.3204	.2742	.2289	.1829	.1332
60	.5671	.4846	.4221	.3686	.3198	.2737	.2285	.1826	.1330
61	.5660	.4836	.4213	.3678	.3192	.2731	.2280	.1822	.1327
62	.5649	.4826	.4204	.3671	.3185	.2725	.2275	.1819	.1325
63	.5637	.4815	.4194	.3662	.3178	.2719	.2270	.1814	.1322
64	.5624	.4803	.4184	.3653	.3170	.2712	.2265	.1810	.1319
65	.5610	.4791	.4173	.3644	.3162	.2705	.2259	.1805	.1316
66	.5596	.4778	.4161	.3633	.3153	.2698	.2253	.1801	.1312
67	.5579	.4764	.4149	.3623	.3144	.2690	.2246	.1795	.1308
68	.5562	.4749	.4136	.3611	.3134	.2682	.2239	.1790	.1305
69	.5544	.4733	.4122	.3599	.3123	.2673	.2232	.1784	.1301
70	.5525	.4717	.4108	.3587	.3113	.2664	.2224	.1778	.1297
71	.5505	.4699	.4092	.3573	.3101	.2654	.2216	.1772	.1292
72	.5484	.4680	.4076	.3559	.3089	.2644	.2208	.1765	.1288
73	.5461	.4661	.4059	.3545	.3076	.2633	.2199	.1758	.1283
74	.5437	.4641	.4042	.3529	.3063	.2622	.2190	.1751	.1278
75	.5412	.4619	.4023	.3513	.3049	.2610	.2180	.1743	.1273
76	.5386	.4597	.4004	.3496	.3035	.2598	.2170	.1736	.1267
77	.5358	.4573	.3983	.3479	.3020	.2585	.2159	.1727	.1261
78	.5329	.4548	.3962	.3461	.3004	.2572	.2148	.1719	.1256
79	.5299	.4523	.3940	.3442	.2988	.2558	.2137	.1710	.1250
80	.5267	.4496	.3917	.3422	.2971	.2544	.2125	.1700	.1243
81	.5234	.4468	.3893	.3401	.2953	.2529	.2113	.1691	.1237
82	.5199	.4439	.3868	.3380	.2935	.2513	.2100	.1681	.1230
83	.5163	.4409	.3842	.3358	.2916	.2497	.2087	.1670	.1223
84	.5126	.4378	.3816	.3335	.2896	.2480	.2073	.1660	.1216
85	.5088	.4346	.3788	.3311	.2875	.2463	.2058	.1649	.1208
86	.5048	.4313	.3760	.3286	.2854	.2445	.2044	.1637	.1201
87	.5008	.4279	.3731	.3261	.2832	.2426	.2028	.1625	.1193
88	.4966	.4244	.3700	.3234	.2810	.2407	.2012	.1613	.1184
89	.4924	.4208	.3669	.3207	.2786	.2387	.1996	.1600	.1176

Note: This table gives the powers of the M.D.L.R.T. and modified L.R.T. since they are equivelant to the L.R.T. for $\beta^{*}=2\pi$

TABLE 5

The power function of the M.D.L.R.T. for the case $\beta^{*}=\pi/6$

β	value of Δ .1	.2	.3	.4	.5	.6	.7	.8	.9
1	.5228	.4521	.3980	.3509	.3073	.2653	.2233	.1797	.1314
2	.5263	.4550	.4005	.3531	.3092	.2668	.2245	.1806	.1320
3	.5296	.4578	.4029	.3551	.3109	.2682	.2257	.1815	.1325
4	.5328	.4604	.4051	.3570	.3125	.2696	.2267	.1823	.1331
5	.5358	.4629	.4072	.3588	.3140	.2708	.2277	.1831	.1335
6	.5387	.4652	.4092	.3605	.3154	.2720	.2287	.1838	.1340
7	.5414	.4675	.4110	.3620	.3167	.2731	.2296	.1844	.1344
8	.5440	.4695	.4128	.3635	.3180	.2741	.2304	.1850	.1348
9	.5464	.4715	.4144	.3649	.3191	.2750	.2311	.1856	.1351
10	.5487	.4734	.4159	.3661	.3202	.2759	.2318	.1861	.1354
11	.5509	.4751	.4174	.3673	.3211	.2767	.2324	.1865	.1357
12	.5529	.4767	.4187	.3684	.3220	.2774	.2330	.1869	.1360
13	.5549	.4783	.4199	.3694	.3228	.2780	.2335	.1873	.1362
14	.5567	.4797	.4211	.3704	.3236	.2786	.2339	.1876	.1364
15	.5584	.4810	.4222	.3712	.3242	.2792	.2343	.1879	.1366
16	.5600	.4823	.4231	.3720	.3249	.2796	.2347	.1882	.1368
17	.5614	.4834	.4240	.3727	.3254	.2800	.2350	.1884	.1369
18	.5628	.4845	.4248	.3733	.3259	.2804	.2352	.1885	.1370
19	.5641	.4854	.4255	.3739	.3263	.2807	.2354	.1887	.1371
20	.5653	.4863	.4263	.3744	.3266	.2809	.2356	.1888	.1371
21	.5664	.4871	.4269	.3748	.3269	.2811	.2357	.1888	.1372
22	.5674	.4878	.4274	.3752	.3272	.2813	.2358	.1888	.1372
23	.5683	.4885	.4278	.3755	.3274	.2814	.2358	.1888	.1371
24	.5691	.4891	.4282	.3757	.3275	.2814	.2358	.1888	.1371
25	.5698	.4896	.4286	.3759	.3276	.2814	.2357	.1887	.1370
26	.5705	.4900	.4288	.3760	.3276	.2814	.2356	.1886	.1369
27	.5711	.4904	.4290	.3761	.3276	.2813	.2355	.1885	.1368
28	.5716	.4907	.4292	.3762	.3275	.2812	.2353	.1883	.1367
29	.5720	.4909	.4293	.3761	.3274	.2810	.2351	.1881	.1366
30	.5724	.4911	.4293	.3760	.3273	.2808	.2349	.1879	.1364

TABLE 6

The power function of the M.D.L.R.T. for the case $\beta^{*}=\pi/3$

β	value of Δ .1	.2	.3	.4	.5	.6	.7	.8	.9
1	.5081	.4369	.3828	.3361	.2931	.2521	.2114	.1697	.1242
2	.5124	.4405	.3859	.3387	.2954	.2540	.2130	.1709	.1250
3	.5165	.4439	.3888	.3413	.2976	.2558	.2145	.1720	.1257
4	.5204	.4472	.3917	.3437	.2997	.2576	.2159	.1731	.1264
5	.5242	.4504	.3944	.3461	.3017	.2592	.2173	.1741	.1270
6	.5279	.4534	.3970	.3483	.3036	.2608	.2185	.1751	.1277
7	.5314	.4563	.3995	.3505	.3054	.2624	.2198	.1761	.1283
8	.5347	.4591	.4019	.3525	.3072	.2638	.2210	.1770	.1289
9	.5379	.4618	.4041	.3544	.3088	.2652	.2221	.1778	.1294
10	.5410	.4643	.4063	.3563	.3104	.2665	.2231	.1786	.1299
11	.5439	.4668	.4084	.3580	.3119	.2677	.2241	.1794	.1304
12	.5466	.4691	.4103	.3597	.3133	.2689	.2251	.1801	.1309
13	.5492	.4712	.4122	.3613	.3146	.2700	.2260	.1808	.1314
14	.5517	.4733	.4139	.3628	.3159	.2711	.2268	.1814	.1318
15	.5540	.4753	.4156	.3642	.3171	.2721	.2276	.1820	.1322
16	.5562	.4771	.4172	.3655	.3182	.2730	.2284	.1826	.1326
17	.5583	.4789	.4187	.3668	.3192	.2738	.2290	.1831	.1329
18	.5603	.4805	.4201	.3680	.3202	.2747	.2297	.1836	.1332
19	.5621	.4821	.4214	.3691	.3211	.2754	.2303	.1841	.1335
20	.5638	.4835	.4226	.3701	.3220	.2761	.2309	.1845	.1338
21	.5655	.4849	.4238	.3711	.3228	.2768	.2314	.1849	.1341
22	.5670	.4862	.4248	.3720	.3235	.2774	.2318	.1852	.1343
23	.5684	.4874	.4258	.3728	.3242	.2779	.2323	.1855	.1345
24	.5697	.4885	.4268	.3736	.3249	.2784	.2327	.1858	.1347
25	.5709	.4895	.4276	.3743	.3254	.2789	.2330	.1861	.1348
26	.5720	.4905	.4284	.3749	.3260	.2793	.2333	.1863	.1350
27	.5731	.4913	.4291	.3755	.3264	.2797	.2336	.1865	.1351
28	.5740	.4921	.4298	.3760	.3269	.2800	.2338	.1867	.1352
29	.5749	.4928	.4304	.3765	.3272	.2803	.2340	.1868	.1353
30	.5757	.4935	.4309	.3769	.3275	.2805	.2342	.1869	.1354
31	.5764	.4941	.4314	.3773	.3278	.2807	.2343	.1870	.1354
32	.5771	.4946	.4318	.3776	.3280	.2809	.2344	.1870	.1354
33	.5776	.4950	.4321	.3779	.3282	.2810	.2345	.1871	.1354
34	.5781	.4954	.4324	.3781	.3283	.2810	.2345	.1871	.1354
35	.5785	.4957	.4326	.3782	.3284	.2811	.2345	.1870	.1354
36	.5789	.4960	.4328	.3783	.3285	.2811	.2345	.1870	.1353
37	.5792	.4961	.4329	.3783	.3285	.2810	.2344	.1869	.1353
38	.5794	.4963	.4329	.3783	.3284	.2809	.2343	.1868	.1352
39	.5795	.4963	.4329	.3783	.3283	.2808	.2342	.1866	.1351
40	.5796	.4963	.4329	.3782	.3282	.2806	.2340	.1865	.1349
41	.5796	.4963	.4328	.3780	.3280	.2804	.2338	.1863	.1348
42	.5796	.4962	.4326	.3778	.3277	.2802	.2335	.1861	.1346
43	.5795	.4960	.4324	.3776	.3275	.2799	.2333	.1858	.1344
44	.5793	.4957	.4321	.3773	.3272	.2796	.2330	.1856	.1342
45	.5790	.4954	.4317	.3769	.3268	.2792	.2326	.1853	.1340

TABLE 6, cont.

β	.1	.2	.3	.4	.5	.6	.7	.8	.9
	value of Δ								
46	.5787	.4951	.4314	.3765	.3264	.2788	.2323	.1850	.1338
47	.5783	.4946	.4309	.3760	.3259	.2784	.2319	.1846	.1335
48	.5779	.4941	.4304	.3755	.3254	.2779	.2314	.1842	.1333
49	.5774	.4936	.4298	.3750	.3249	.2774	.2310	.1838	.1330
50	.5768	.4930	.4292	.3743	.3243	.2769	.2305	.1834	.1327
51	.5761	.4923	.4285	.3737	.3237	.2763	.2299	.1830	.1323
52	.5754	.4915	.4278	.3730	.3230	.2756	.2294	.1825	.1320
53	.5746	.4907	.4270	.3722	.3222	.2750	.2288	.1820	.1316
54	.5737	.4898	.4261	.3713	.3215	.2742	.2281	.1815	.1312
55	.5728	.4889	.4252	.3705	.3206	.2735	.2275	.1809	.1308
56	.5717	.4878	.4242	.3695	.3197	.2727	.2267	.1803	.1304
57	.5706	.4867	.4231	.3685	.3188	.2718	.2260	.1797	.1300
58	.5694	.4855	.4220	.3674	.3178	.2709	.2252	.1790	.1295
59	.5681	.4843	.4208	.3663	.3168	.2700	.2244	.1784	.1291
60	.5668	.4829	.4195	.3651	.3157	.2690	.2235	.1777	.1286

TABLE 7

The power function of the modified L.R.T. for the case $\beta^{*}=\pi/6$

β	value of Δ .1	.2	.3	.4	.5	.6	.7	.8	.9
1	.5300	.4596	.4055	.3582	.3143	.2717	.2291	.1845	.1348
2	.5328	.4619	.4075	.3599	.3157	.2729	.2300	.1853	.1353
3	.5355	.4641	.4094	.3615	.3171	.2741	.2309	.1859	.1357
4	.5380	.4662	.4111	.3630	.3183	.2751	.2317	.1865	.1361
5	.5403	.4681	.4127	.3643	.3194	.2760	.2325	.1870	.1364
6	.5425	.4698	.4141	.3655	.3204	.2768	.2331	.1875	.1367
7	.5445	.4714	.4154	.3666	.3213	.2776	.2337	.1880	.1370
8	.5464	.4729	.4166	.3676	.3222	.2782	.2342	.1883	.1372
9	.5482	.4743	.4177	.3685	.3229	.2788	.2346	.1887	.1374
10	.5499	.4755	.4187	.3693	.3235	.2793	.2350	.1889	.1376
11	.5514	.4767	.4196	.3700	.3240	.2797	.2353	.1892	.1377
12	.5529	.4777	.4204	.3706	.3245	.2800	.2356	.1893	.1378
13	.5542	.4787	.4211	.3711	.3249	.2803	.2358	.1895	.1379
14	.5554	.4796	.4218	.3716	.3252	.2805	.2359	.1895	.1380
15	.5566	.4804	.4223	.3720	.3255	.2807	.2360	.1896	.1380
15	.5566	.4804	.4223	.3720	.3255	.2807	.2360	.1896	.1380
16	.5576	.4811	.4228	.3723	.3257	.2808	.2360	.1896	.1380
17	.5585	.4817	.4232	.3725	.3258	.2808	.2360	.1896	.1379
18	.5594	.4822	.4235	.3727	.3258	.2808	.2360	.1895	.1379
19	.5602	.4827	.4237	.3728	.3258	.2807	.2359	.1894	.1378
20	.5609	.4831	.4239	.3728	.3258	.2806	.2357	.1892	.1377
21	.5615	.4834	.4241	.3728	.3257	.2804	.2355	.1890	.1375
22	.5620	.4836	.4241	.3727	.3255	.2802	.2353	.1888	.1374
23	.5625	.4838	.4241	.3726	.3253	.2799	.2350	.1885	.1372
24	.5628	.4839	.4240	.3724	.3250	.2796	.2347	.1882	.1370
25	.5631	.4840	.4239	.3722	.3247	.2793	.2343	.1879	.1367
26	.5634	.4839	.4237	.3719	.3243	.2789	.2339	.1876	.1365
27	.5636	.4839	.4235	.3715	.3239	.2784	.2335	.1872	.1362
28	.5637	.4837	.4232	.3711	.3234	.2779	.2330	.1868	.1359
29	.5637	.4835	.4229	.3707	.3229	.2774	.2325	.1863	.1356
30	.5637	.4833	.4225	.3702	.3224	.2768	.2319	.1858	.1353

TABLE 8

The power function of the modified L.R.T. for the case $\beta^{*}=\pi/3$

β	value of Λ .1	.2	.3	.4	.5	.6	.7	.8	.9
1	.5080	.4368	.3828	.3362	.2933	.2523	.2118	.1702	.1249
2	.5124	.4405	.3860	.3389	.2956	.2543	.2134	.1715	.1257
3	.5165	.4440	.3890	.3415	.2979	.2562	.2149	.1726	.1265
4	.5206	.4474	.3919	.3440	.3000	.2580	.2164	.1738	.1272
5	.5245	.4507	.3947	.3464	.3021	.2597	.2178	.1748	.1279
6	.5282	.4538	.3974	.3488	.3041	.2614	.2192	.1759	.1286
7	.5318	.4568	.4000	.3510	.3060	.2630	.2205	.1769	.1292
8	.5353	.4597	.4024	.3531	.3078	.2645	.2217	.1778	.1298
9	.5386	.4624	.4048	.3551	.3095	.2659	.2229	.1787	.1304
10	.5417	.4651	.4070	.3570	.3111	.2673	.2240	.1796	.1310
11	.5447	.4676	.4092	.3589	.3127	.2686	.2251	.1804	.1315
12	.5475	.4700	.4112	.3606	.3142	.2699	.2261	.1811	.1320
13	.5502	.4722	.4132	.3623	.3155	.2710	.2270	.1819	.1325
14	.5528	.4744	.4150	.3639	.3169	.2722	.2279	.1826	.1330
15	.5552	.4765	.4168	.3654	.3182	.2732	.2288	.1832	.1334
16	.5575	.4784	.4184	.3668	.3194	.2742	.2296	.1839	.1338
17	.5597	.4802	.4200	.3681	.3205	.2752	.2304	.1844	.1342
18	.5617	.4820	.4215	.3694	.3216	.2760	.2311	.1850	.1346
19	.5636	.4836	.4229	.3706	.3226	.2769	.2317	.1855	.1349
20	.5654	.4851	.4242	.3717	.3236	.2777	.2324	.1860	.1353
21	.5671	.4866	.4254	.3727	.3244	.2784	.2330	.1864	.1355
22	.5687	.4879	.4266	.3737	.3253	.2791	.2335	.1868	.1358
23	.5702	.4892	.4277	.3746	.3260	.2797	.2340	.1872	.1361
24	.5712	.4904	.4287	.3755	.3267	.2803	.2345	.1875	.1363
25	.5728	.4915	.4296	.3763	.3274	.2808	.2349	.1879	.1365
26	.5740	.4925	.4305	.3770	.3280	.2813	.2353	.1881	.1367
27	.5751	.4935	.4313	.3777	.3285	.2817	.2356	.1884	.1369
28	.5761	.4943	.4320	.3783	.3290	.2821	.2359	.1886	.1370
29	.5771	.4951	.4327	.3788	.3295	.2825	.2362	.1888	.1371
30	.5779	.4958	.4333	.3793	.3299	.2828	.2364	.1890	.1372
31	.5787	.4965	.4338	.3798	.3302	.2831	.2366	.1891	.1373
32	.5794	.4971	.4343	.3801	.3305	.2833	.2368	.1892	.1374
33	.5800	.4976	.4347	.3805	.3308	.2835	.2369	.1893	.1374
34	.5806	.4980	.4351	.3808	.3310	.2836	.2370	.1894	.1375
35	.5811	.4984	.4354	.3810	.3312	.2837	.2371	.1894	.1375
36	.5815	.4987	.4356	.3812	.3313	.2838	.2371	.1894	.1375
37	.5818	.4990	.4358	.3813	.3313	.2838	.2371	.1894	.1374
38	.5821	.4992	.4359	.3814	.3314	.2838	.2370	.1893	.1374
39	.5823	.4993	.4360	.3814	.3314	.2838	.2370	.1892	.1373
40	.5825	.4994	.4360	.3813	.3313	.2837	.2369	.1891	.1372
41	.5825	.4994	.4360	.3813	.3312	.2835	.2367	.1890	.1371
42	.5825	.4994	.4359	.3811	.3310	.2834	.2366	.1889	.1370
43	.5825	.4993	.4358	.3810	.3309	.2832	.2364	.1887	.1369
44	.5824	.4991	.4356	.3808	.3306	.2829	.2361	.1885	.1367

TABLE 8, cont.

β	value of Δ								
	.1	.2	.3	.4	.5	.6	.7	.8	.9
45	.5822	.4989	.4353	.3805	.3303	.2827	.2359	.1882	.1366
46	.5819	.4986	.4350	.3802	.3300	.2824	.2356	.1880	.1364
47	.5816	.4982	.4346	.3798	.3296	.2820	.2353	.1877	.1362
48	.5812	.4978	.4342	.3794	.3292	.2816	.2349	.1874	.1359
49	.5808	.4973	.4337	.3789	.3288	.2812	.2345	.1871	.1357
50	.5803	.4968	.4332	.3784	.3283	.2807	.2341	.1867	.1354
51	.5797	.4962	.4326	.3778	.3277	.2802	.2336	.1863	.1352
52	.5790	.4955	.4319	.3771	.3271	.2797	.2331	.1859	.1349
53	.5783	.4948	.4312	.3765	.3265	.2791	.2326	.1855	.1345
54	.5775	.4940	.4304	.3757	.3258	.2784	.2321	.1850	.1342
55	.5766	.4931	.4296	.3749	.3250	.2778	.2315	.1845	.1339
56	.5757	.4922	.4287	.3741	.3243	.2770	.2308	.1840	.1335
57	.5746	.4912	.4277	.3732	.3234	.2763	.2302	.1835	.1331
58	.5735	.4901	.4267	.3722	.3225	.2755	.2295	.1829	.1327
59	.5723	.4889	.4255	.3711	.3216	.2746	.2287	.1823	.1322
60	.5710	.4876	.4244	.3701	.3206	.2737	.2280	.1816	.1318

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VITA

Mr. Bassam Suleiman Abu-Amra was born in Gaza on Nov. 19, 1964. After receiving the General Secondary Education Certificate in 1983, he got admission at Yarmouk University and recieved his Bachelor Degree of Science (Statistics) in 1987 there. He joined the Master's programme in the department of statistics at Yarmouk Universiy in September, 1987. As a graduate student he was working as a teaching assistant in the Department of Statistics.